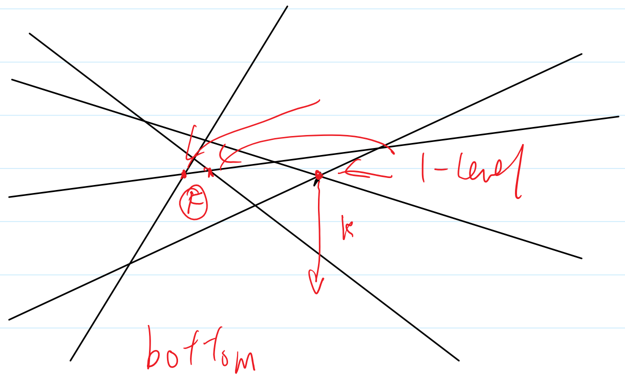
On the Average Complexity of the k-level

Joint work with Stefan Felsner, Manfred Scheucher, Patrick Schnider, Raphael Steiner and Pavel Valtr

Dach

Graz

Patrick



Previous results
on the size of the k-level:

$O(nk^{1/3})$ by Dey

$ne^{\Omega(\sqrt{\log k})}$ by Nivash

median $O(n^{4/3})$ smaller value by averaging

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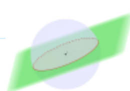
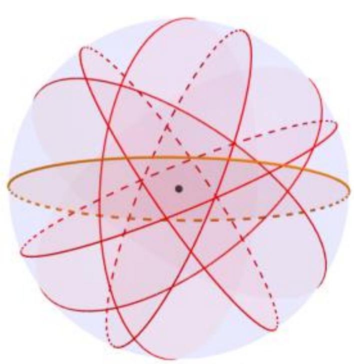
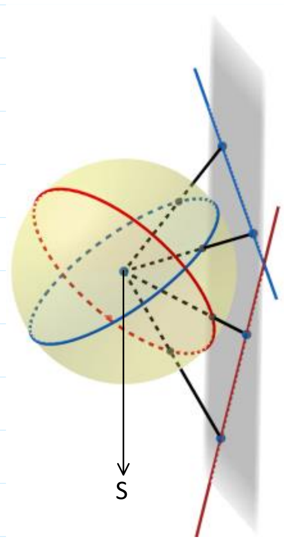
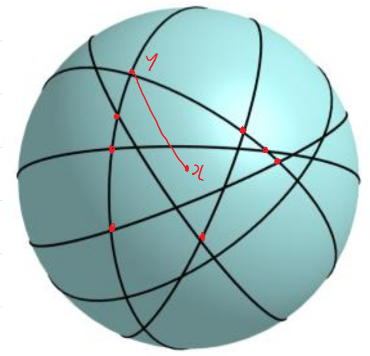
great-circle arrangement

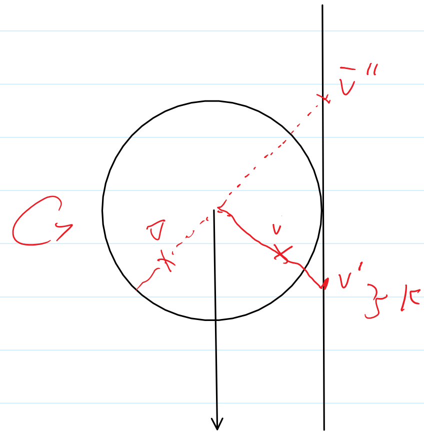






dist(x,y) and k-level of $\mathscr{C}$





$|k\text{-level of } C| = |k\text{-level of } L| + |(n-k-2)\text{-level of } L|$

is it tight?

> **Thm.** Given any arrangement of n great-circles. The expected size of the $(\leq k)$-level from a random cell is $O(k^2)$.

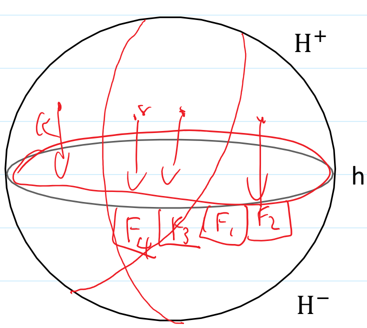
$k \ll n^{3/8}$

k-level

$$\text{Expected size} = \frac{\sum_{cell\ F\ in\ C} |\{ v\ in\ C : \mathrm{dist}(F, v) \leq k\}|}{\#\text{cells}} = \frac{O(k^2 n^2)}{2\binom{n}{2} + 2 = \Theta(n^2)} = O(k^2)$$

Higher order zone theorem : Let $\mathscr{L}$ be a simple arrangement of n lines in R^2 and h in $\mathscr{L}$.
The $(\leq k)$-zone of h contains O(kn).

Fix H^-

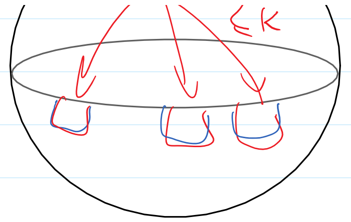


$$\sum_{cell\ F\ in\ C} |\{ v\ in\ C : \mathrm{dist}(F, v) \leq k\}|$$

$$\leq \underbrace{\sum_{H^-}}_{O(n)} \underbrace{\sum_{boundary\ cell\ F\ in\ H^-}}_{O(n)} \underbrace{|\{ v\ in\ H^+ : \mathrm{dist}(F, v) \leq k\}|}_{O(kn)}$$

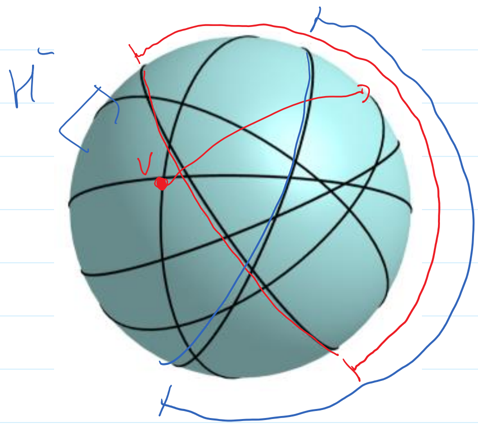
Fix H^-, v

$$= \underbrace{\sum_{H^-}}_{O(n)} \underbrace{\sum_{v\ in\ H^+}}_{O(kn)} \underbrace{|\{ bdy\ F\ in\ H^- : \mathrm{dist}(F, v) \leq k\}|}_{O(k)} = O(k^2 n^2)$$

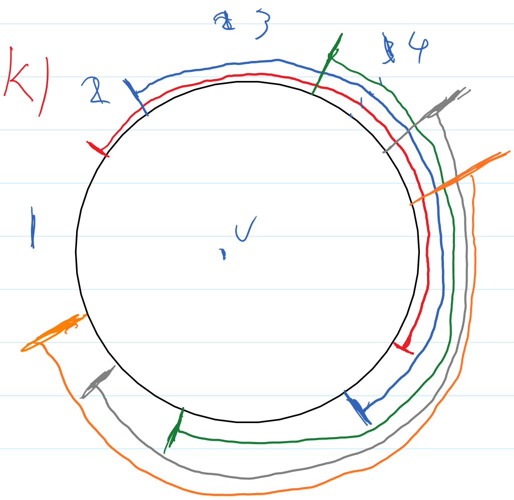


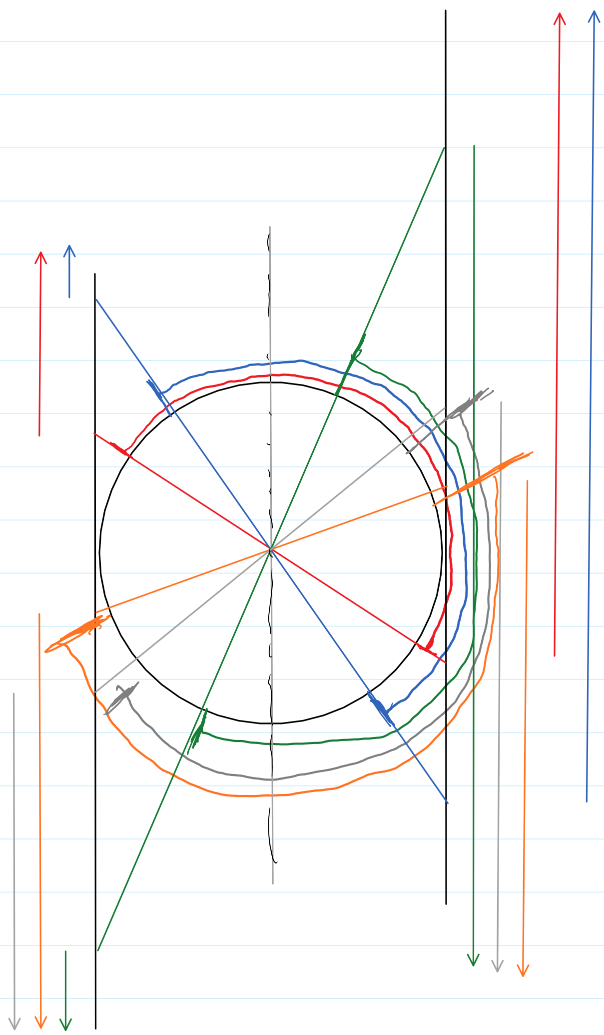
$O(n)$ vv $O(kn)$

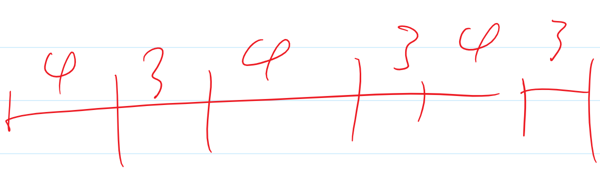
$O(K) = O(K^2 n^2)$



$O(k)$

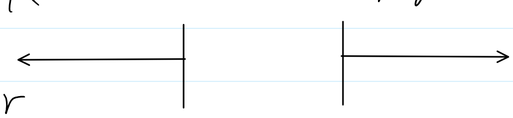






left

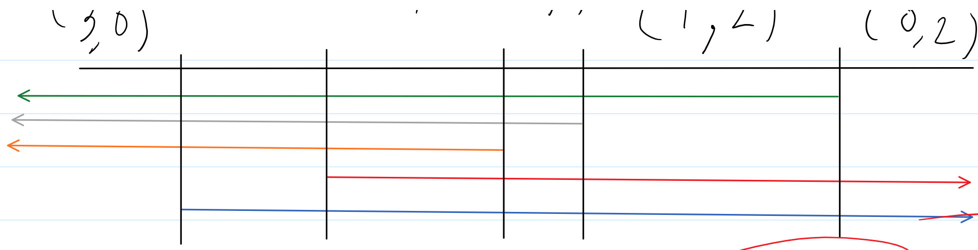
right

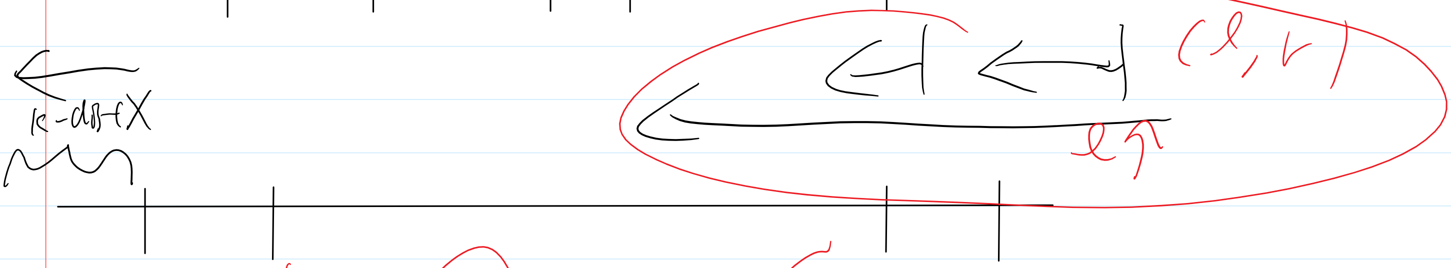


\# (l, r) s.t. $l + r \le k$

l r

(3,0) (3,1) (3,2) (2,2) (1,2) (0,2)





(x, y) $\quad$? $\quad$ (x', y')

$x + y \leq k-1$ $\quad$ $x' + y' \leq k-1$

largest left-most $\quad$ largest right-most

\# intervals from $(x,y) \to (x', y')$

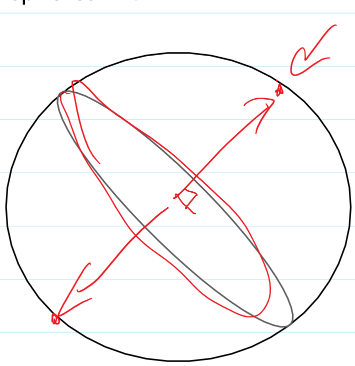
$= x - x' + y' - y + 1$

$\leq 2k - 1$

l $\quad$ $l-1$ $\quad$ or r $\quad$ $r+1$ $\quad$ $x \leq k-1$, $y' \leq k-1$

Random great-(d-1)-spheres in S^d

random



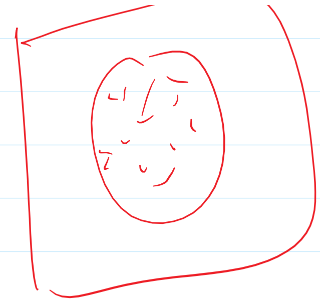
Thm: Let d ≥ 2 be fixed. In an arrangement of n great-(d − 1)-spheres chosen uniformly at random on the unit sphere S^d (embedded in R^{d+1}),

the expected size of the k-level is $\Theta(k^{d-1})$.

size k-level

Previous results on arrangement in R^d: $\Omega(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - 1}) \leq f_k^{(d)}(n) \leq O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - c_d})$

arrangement in R^d:

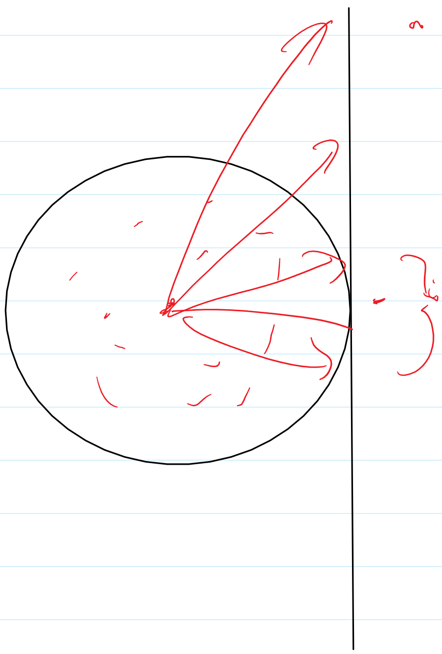


Previous results on random points in R^2:

points sampled uniform at random in a convex shape K

K is a disk, the convex hull (0-level) has expected size is $O(n^{1/3})$

K is a convex polygon with h sides, the expected size is O(h log n)



less

h log n