Mittagsseminar

Freie Universität Berlin

The ε-net lemma and applications to the wet part

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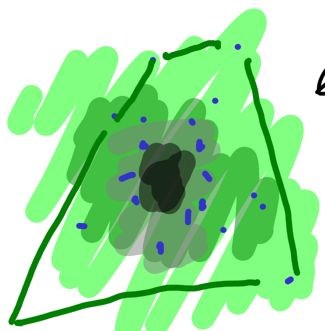
Komlós, Pach, Woeginger 1992; [Pach & Agarwal 1995]

[Haussler, Welzl 1987; Vapnik & Chervonenkis]

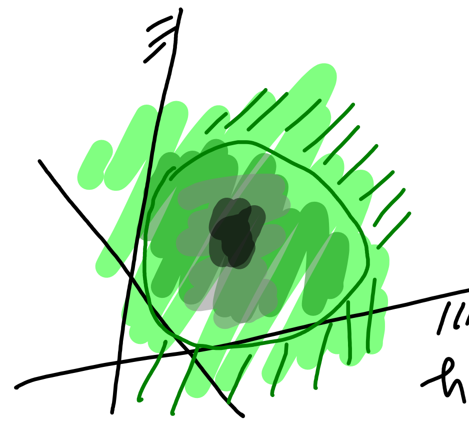


a probability distribution in the plane

The wet part

$$W_\varepsilon := \bigcup \{ \text{halfspace } h \mid \mu(h) \le \varepsilon \}$$

$$w(\varepsilon) := \mu(W_\varepsilon)$$



THM:

$$\frac{1}{e}\, n \boxed{w\left(\tfrac{1}{n}\right)} \le E\left(\#\text{hull vertices of } X_n\right) \le n \cdot w\left(\underbrace{(d+2)}_{\text{dim.}}\frac{\ln n}{n}\right) + o_d(1)$$

X is an ε-net $\iff$

$$\forall \text{ halfspaces } h: \mu(h) \ge \varepsilon \Rightarrow h \cap X \ne \emptyset$$

$$X \ \varepsilon\text{-net} \Rightarrow \operatorname{conv}(X) \supseteq \mathbb{R}^d - W_\varepsilon$$

Lemma: Haussler Welzl 1987

X_s random sample $|X_s| = s$

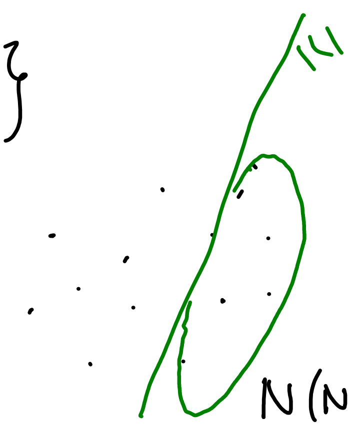
Pick $N > s$

$$\mathrm{Prob}\left[X_s \text{ is NOT an } \varepsilon\text{-net}\right] \le 2\, \underbrace{\pi(N)} \cdot \left(1 - \frac{s}{N}\right)^{(N-s)\varepsilon - 1}$$

$\pi(N) =$ shatter function

$$\max\left\{ \#\{U \cap h \mid \text{halfsp } h\} \;\middle|\; |U| = N \right\}$$

$\le N^d$



$N(N-1)+2$

$N = s \log s$ ..

$s = \frac{1}{\varepsilon} d \log N \Rightarrow \Pr < 1$

$\frac{1}{\varepsilon}(d+2) \log N \Rightarrow \quad \le \frac{1}{n}$

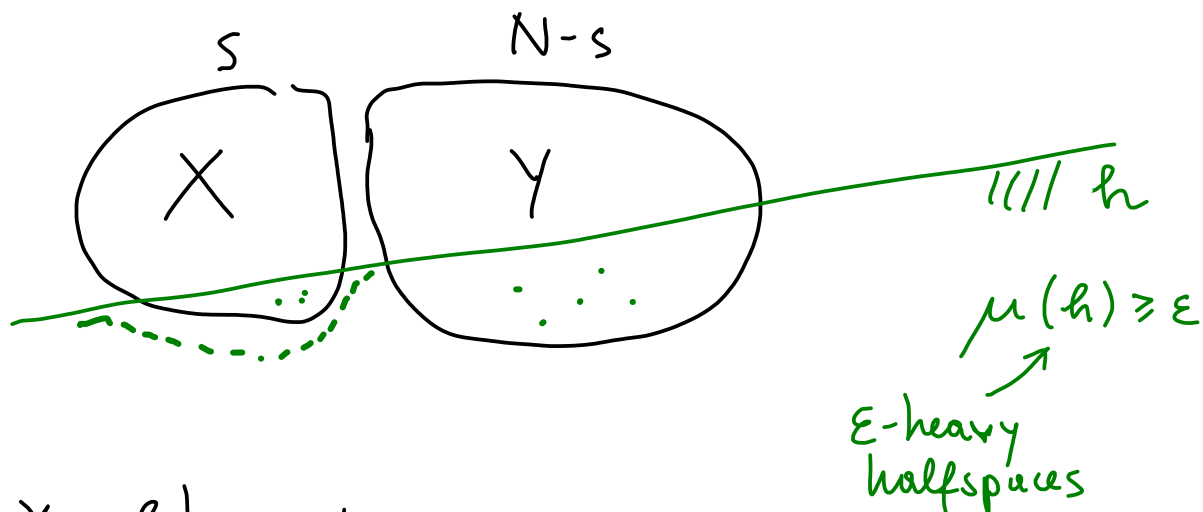
$\frac{1}{\varepsilon}(2d) \log n \quad \le$ exp. small

$\boxed{\text{Lemma:}}$ Haussler Welzl 1987

X_s random sample $|X_s| = s$

Pick $N > s$

$$\boxed{\begin{aligned} &\mathrm{Prob}\left[X_s \text{ is NOT an } \varepsilon\text{-net}\right] \\ &\qquad \le 2\, \underbrace{\pi(N)}_{} \cdot \left(1 - \frac{s}{N}\right)^{(N-s)\varepsilon - 1} \end{aligned}}$$



$\#_h(X) := |X \cap h|$ random var.
binomial

fixed

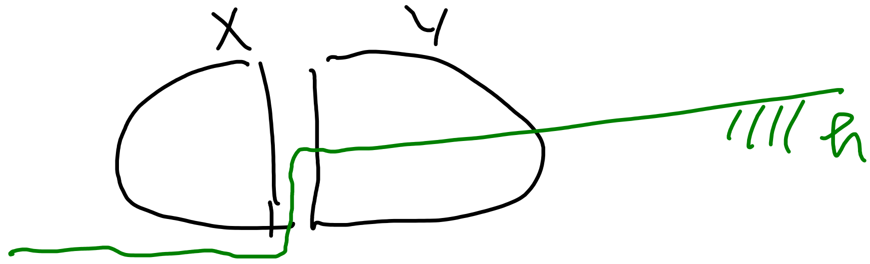
mean $= s \cdot \mu(h)$

$|\text{median} - \text{mean}| \le 1$ $\quad m_h$ median for Y

$$\Pr(\#_h(Y) < m_h) \le \frac{1}{2} \le P[\#_h(Y) \le m_h]$$

$(N-s)\mu(h) \pm 1$

$Z = X \cup Y$

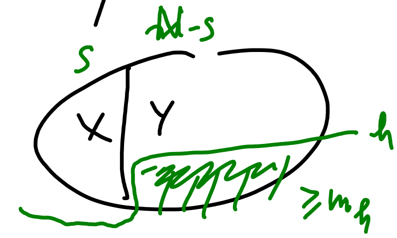


h is skew: $\#_h(X) = 0 \;\wedge\; \#_h(Y) \geq m_h$

$$\ldots \geq \Pr[\text{Exist } \varepsilon\text{-heavy } h \text{ skew for } X, Y]$$

$$\geq \underbrace{\Pr[\exists\, \varepsilon\text{-heavy } h : \#_h(X) = 0]}_{X \text{ not } \varepsilon\text{-net.}} \times \min_{\varepsilon\text{heavy } h} \underbrace{\Pr[\#_h(Y) \geq m_h]}_{\geq \frac{1}{2}}$$

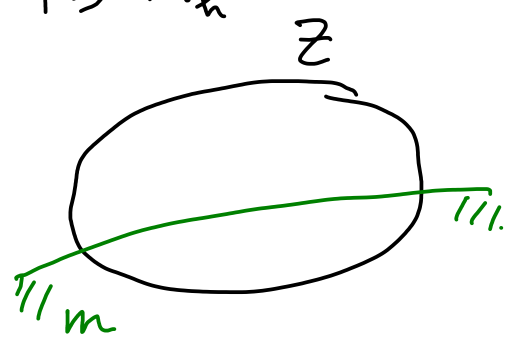
$Z = X \cup Y$ $\boxed{\text{fix } Z}$ partition $Z = X \cup Y$ randomly



$\boxed{\text{fix } h}$ $\Pr[h \text{ skew for } X, Y]$

$$= \begin{cases} \dfrac{\binom{N-m}{s}}{\binom{N}{s}} & \text{if } m = |Z \cap h| \geq m_h \\ 0, & \text{otherwise} \end{cases}$$

$\leftarrow$ choices for X

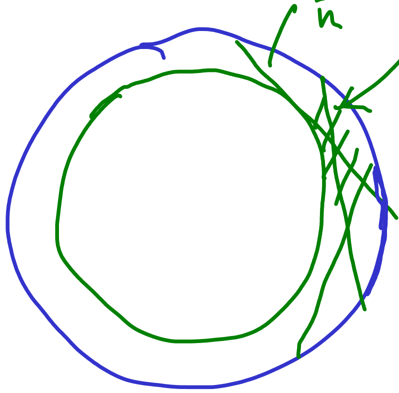


$$\leq \left(1 - \frac{s}{N}\right)^{m_h} \quad m_h \geq (N-s)\underbrace{\mu(h)}_{\geq \varepsilon} - 1$$

variable h: depends on $\underbrace{h \cap Z}_{\leq \pi(N)}$ $\leftarrow$ fixed. $\leftarrow$ random Z

$\frac{1}{n}$

$\mu(W_{\frac{1}{n}}) \sim \frac{1}{n^{2/3}}$



$E(\#\text{hull})$

$\sim n^{1/3}$, $\cancel{n^{2/3} ?}$

$\sim \log n$

this part dominates