Kubische B-Splines

$$N_i^2(t) = \begin{cases} f_0^2(t-i) & i \le t \le i+1 \\ f_1^2(t-i-1) & i+1 \le t \le i+2 \\ f_2^2(t-i-2) & i+2 \le t \le i+3 \end{cases} \qquad \begin{aligned} f_0^2(t) &= t^2/2 \\ f_1^2(t) &= \tfrac{1}{2}(-2t^2+2t+1) \\ f_2^2(t) &= \tfrac{1}{2}(1-t)^2 = \tfrac{1}{2}(t^2-2t+1) \end{aligned}$$

$$\boxed{N_i^3(t) = N_i^2(t) \cdot \frac{t-i}{3} + N_{i+1}^2(t) \cdot \frac{i+4-t}{3}}$$

$$N_i^3(t) = \begin{cases} f_0^3(t') = f_0^2(t') \cdot \frac{t'}{3}, & i \le t \le i+1 \quad (t' = t-i) \\ f_1^3(t') = f_1^2(t') \cdot \frac{t'+1}{3} + f_0^2(t') \cdot \frac{3-t'}{3}, & i+1 \le t \le i+2 \quad (t' = t-i-1) \\ f_2^3(t') = f_2^2(t') \frac{t'+2}{3} + f_1^2(t') \frac{2-t'}{3}, & i+2 \le t \le i+3 \quad (t' = t-i-2) \\ f_3^3(t') = \qquad f_2^2(t') \frac{1-t'}{3}, & i+3 \le t \le i+4 \quad (t' = t-i-3) \end{cases}$$

$$f_0^3(t) = \frac{1}{6} \cdot t^2 \cdot t = \boxed{1/6 \cdot t^3} \qquad f_3^3(t) = \boxed{\frac{1}{6}(1-t)^3}$$

$$f_1^3(t) = \frac{1}{6}\left[(-2t^2+2t+1)(t+1) + t^2 \cdot (3-t)\right]$$

$$\frac{1}{6}\left[-2t^3+2t^2+t-2t^2+2t+1+3t^2-t^3\right] = \boxed{\frac{1}{6}(-3t^3+3t^2+3t+1)}$$

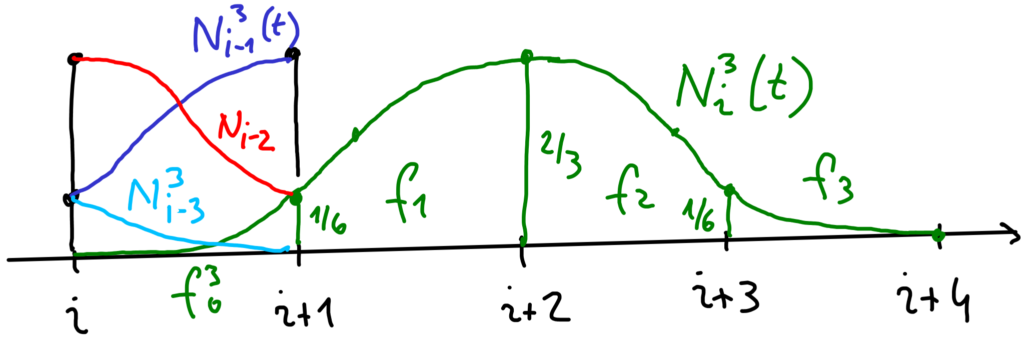
$$f_2^3(t) = \frac{1}{6}\left[\underbrace{(1-t)^2}_{1-2t+t^2}(t+2) + (-2t^2+2t+1)(2-t)\right]$$

$$= \frac{1}{6}\left[t-2t^2+t^3+2-4t+2t^2-4t^2+4t+2+2t^3-2t^2-t\right] = \boxed{\frac{1}{6}(3t^3-6t^2+4)}$$

$$\begin{pmatrix} f_3^3(t) \\ f_2^3(t) \\ f_1^3(t) \\ f_0^3(t) \end{pmatrix} = \frac{1}{6} \underbrace{\begin{pmatrix} -1 & +3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{\text{Basismatrix } A \text{ für kubische B-Splines}} \begin{pmatrix} t^3 \\ t^2 \\ t \\ 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} f_k^3(0)\end{pmatrix}\begin{pmatrix} f_k^3(1)\end{pmatrix}\right] = A\left[\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right] = \left[\begin{pmatrix}1/6\\4/6\\1/6\\0\end{pmatrix}\begin{pmatrix}0\\1/6\\4/6\\1/6\end{pmatrix}\right]$$

$$A = \frac{1}{6}\begin{pmatrix} -1 & +3 & -3 & 1\\ 3 & -6 & 0 & 4\\ -3 & 3 & 3 & 1\\ 1 & 0 & 0 & 0\end{pmatrix}$$

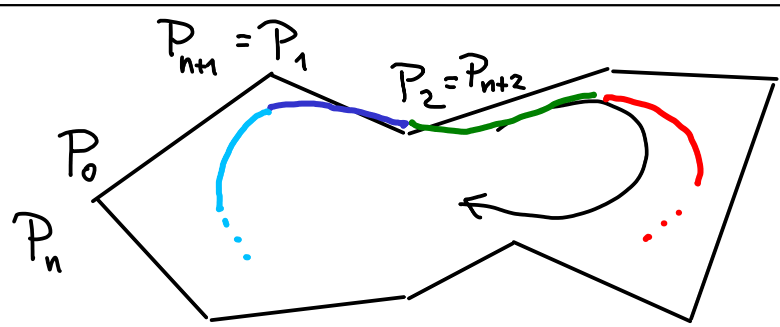


$$\left[f_k'(t), f_k'(0), f_k'(1)\right] = A\begin{pmatrix} 3t^2 & 0 & 3\\ 2t & 0 & 2\\ 1 & 1 & 1\\ 0 & 0 & 0\end{pmatrix} = \frac{1}{6}\begin{pmatrix} \cdot & -3 & 0\\ \cdot & 0 & -3\\ \cdot & 3 & 0\\ \cdot & 0 & 3\end{pmatrix}$$

$$\left[f_k''(t), f_k''(0), f_k''(1)\right] = A\begin{pmatrix} 6t & 0 & 6\\ 2 & 2 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \frac{1}{6}\begin{pmatrix} * & 6 & 0\\ * & -12 & 6\\ * & 6 & -12\\ * & 0 & 6\end{pmatrix} \quad C^2\text{-stetig}$$

$$\boxed{\text{B-Splines vom Grad } d \text{ mit } t_1 < t_2 < t_3 < \ldots \text{ sind } C^{d-1}\text{-stetig.}}$$

Geschlossene B-Splines



Offene B-Splines mit Anfangspunkt und Endpunkt $(d=3)$

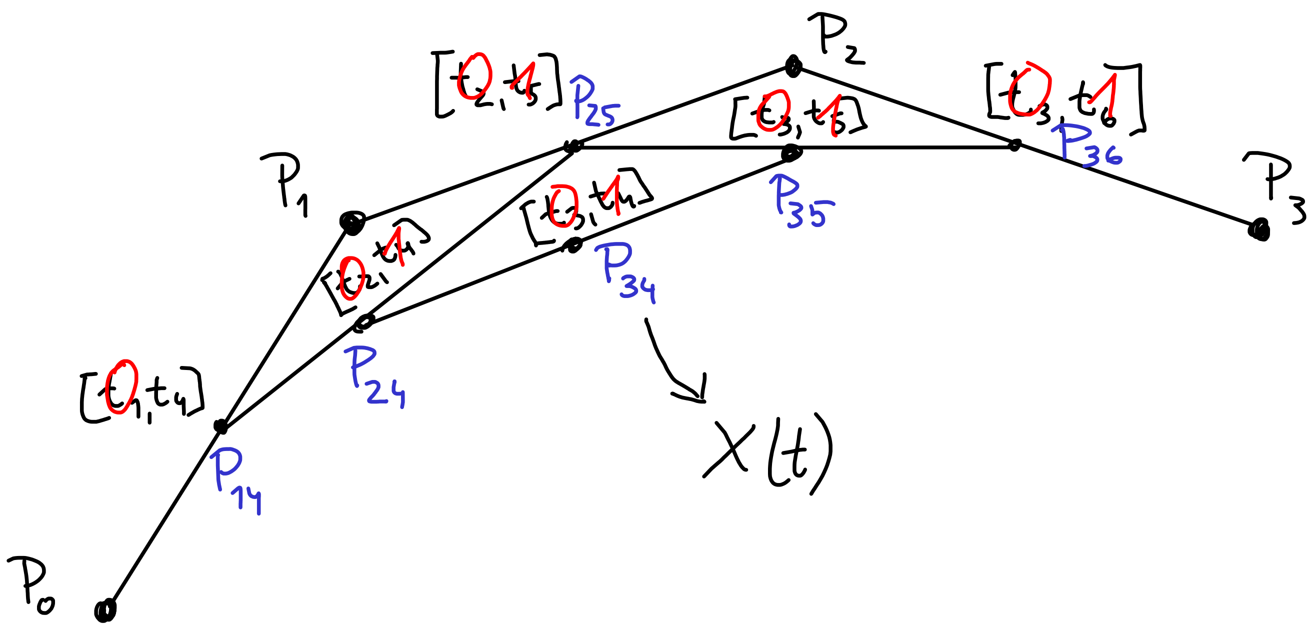
$$\begin{array}{ccccccccccccc} t_0 = & t_1 = & t_2 = & t_3 < & t_4 & \ldots & t_n < & t_{n+1} = & t_{n+2} = & t_{n+3} = & t_{n+4}\\ 0 & 0 & 0 & 0 & 1 & 2 \ldots & n-3 & n-2 & n-2 & n-2 & n-2 \end{array}$$

$$\underbrace{0 \quad 0 \quad 0}_{d \text{ Stück}} \qquad \underbrace{n-2 \quad n-2 \quad n-2}_{d \text{ Stück}}$$

Bézier-Splines als Spezialfall $(d=3)$ $n=3,\quad P_0 P_1 P_2 P_3$

$$\begin{array}{cccccccc} t_0 = & t_1 = & t_2 = & t_3 < & t_4 = & t_5 = & t_6 = & t_7 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}$$

$$\underbrace{0\quad 0\quad 0}_{d\ \text{Stück}} \qquad \underbrace{1\quad 1\quad 1}_{d\ \text{Stück}}$$



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mehrfache Knoten $\quad \ldots < t_i = t_{i+1} = \ldots = t_{i+k} < \ldots$

$\boxed{\text{k+1 gleiche Knoten reduzieren die Glattheit von } C^{d-1} \text{ auf } C^{d-1-k}.}$

mehrfache Kontrollpunkte $\quad P_i = P_{i+1} = \ldots = P_{i+k}$

$\boxed{\text{Eine Häufung von Kontrollpunkten führt zwangsläufig zu geometrischen Singularitäten.}}$

NURBS = non-uniform rational B-splines

- Kreise und Kegelschnitte exakt
- projektive Transformationen