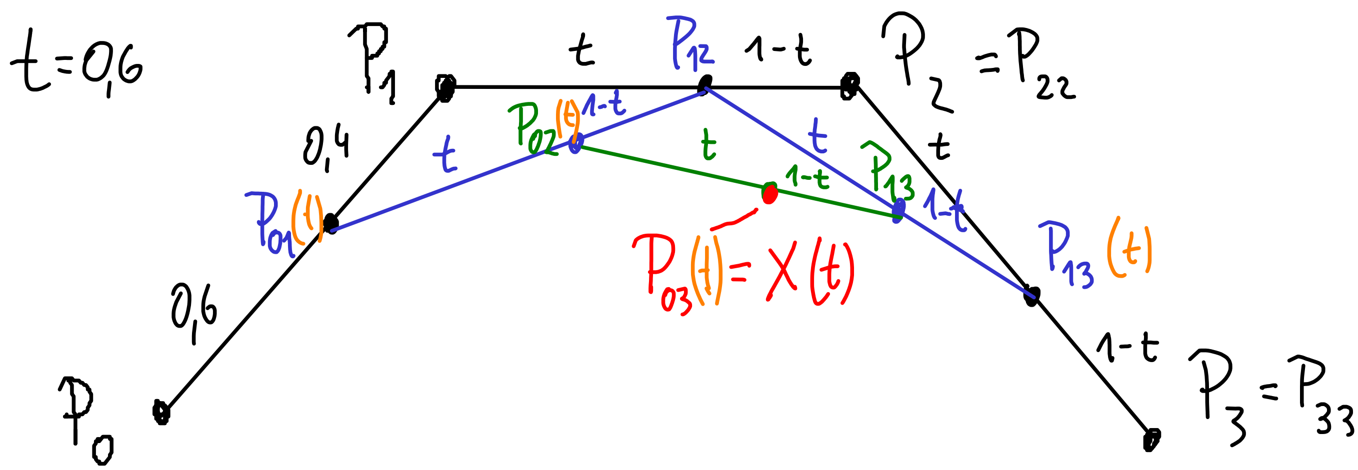
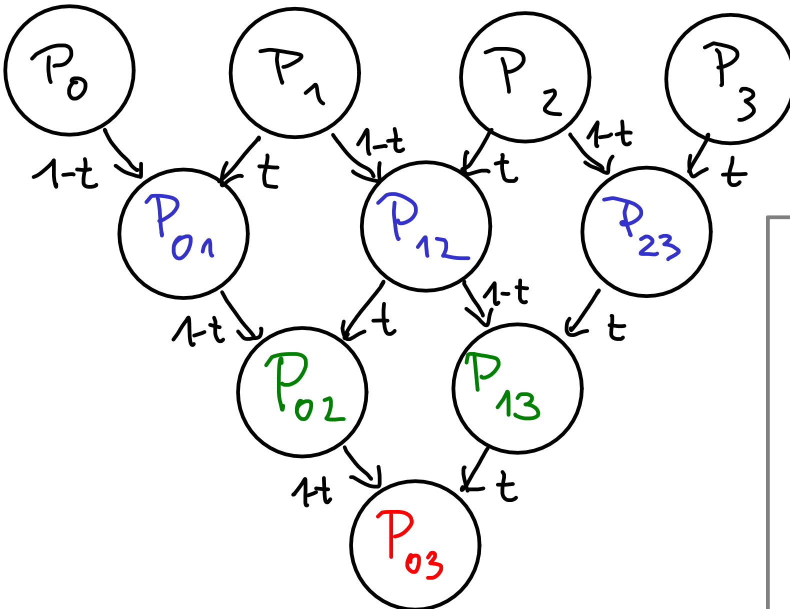
Bézier-Splines, Algorithmus von de Casteljau

Kontrollpunkte $P_0, P_1, \ldots, P_d \in \mathbb{R}^2$ oder $\mathbb{R}^3$

Bernsteinpolynome $B_i^d(t) := \binom{d}{i} t^i (1-t)^{d-i}, 0 \le t \le 1$

$$\boxed{X(t) = \sum_{i=0}^{d} B_i^d(t)\, P_i}$$





$P_{ij}(t)$ = Bézier-Kurve vom Grad $j-i$ mit Kontrollpunkten $P_i, P_{i+1}, \dots, P_j$

vollständige Induktion: $P_{i,i+1} := t\, P_{i+1} + (1-t)\, P_i$ ✓

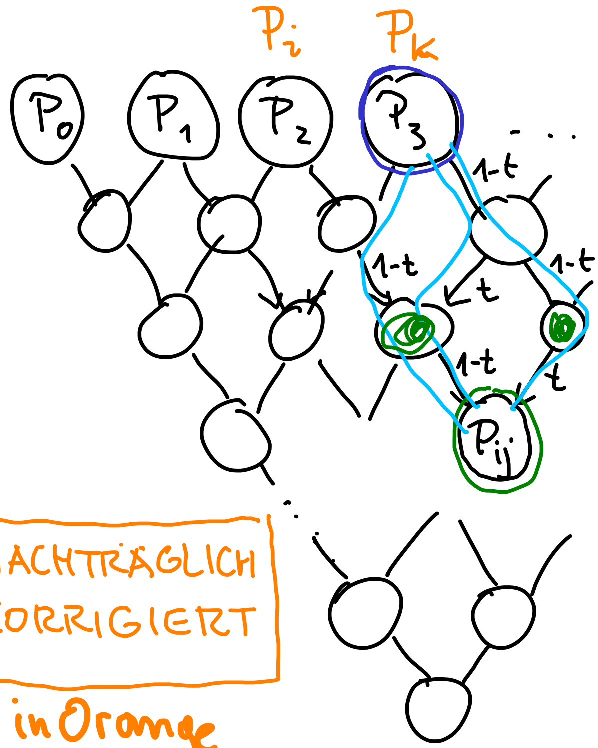
$j \geq i+2$: $P_{i,j} := t\, P_{i+1,j} + (1-t)\, P_{i,j-1}$

$$= t \cdot \sum_{k=i+1}^{j} B^{j-i-1}_{k-(i+1)}(t)\, P_k + (1-t) \sum_{k=i}^{j-1} B^{j-i-1}_{k-i}(t)\, P_k$$

$$= \underbrace{(1-t) B_0^{j-1-i}(t)}_{B_0^{j-i}(t)} P_i + \sum_{k=i+1}^{j-1} \underbrace{\left[t\, B^{j-i-1}_{k-i-1}(t) + (1-t)\, B^{j-i-1}_{k-i}(t) \right]}_{B^{j-i}_{k-i}(t)} \cdot P_k + \underbrace{t\, B^{j-1-i}_{j-i-1}(t)}_{B^{j-i}_{j-i}(t)} P_j$$

[$B_i^n(t) = tB_{i-1}^{n-1}(t) + (1-t)B_i^{n-1}(t)$
(Übung 43d)]

$$= \sum_{k=i}^{j} B^{j-i}_{k-j}(t)\, P_k \qquad \square$$



NACHTRÄGLICH KORRIGIERT

in Orange

HIER STIMMT'S AUSNAHMSWEISE

$$B^{j-i}_{k-j}(t) = \binom{j-i}{k-\cancel{j}\,i}\cdot(1-t)^{j-k}\,t^{k-i}$$

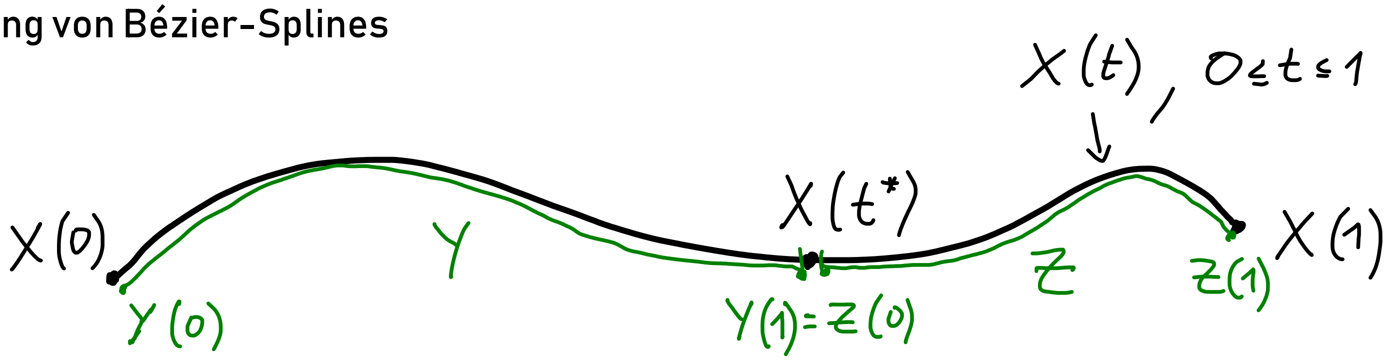
$P_{ij} = \sum_k P_k \cdot$ (Summe über alle Wege von P_k nach P_{ij} [Produkt der Kantengewichte])

$(1-t)^{j-k}\,t^{k-i}$

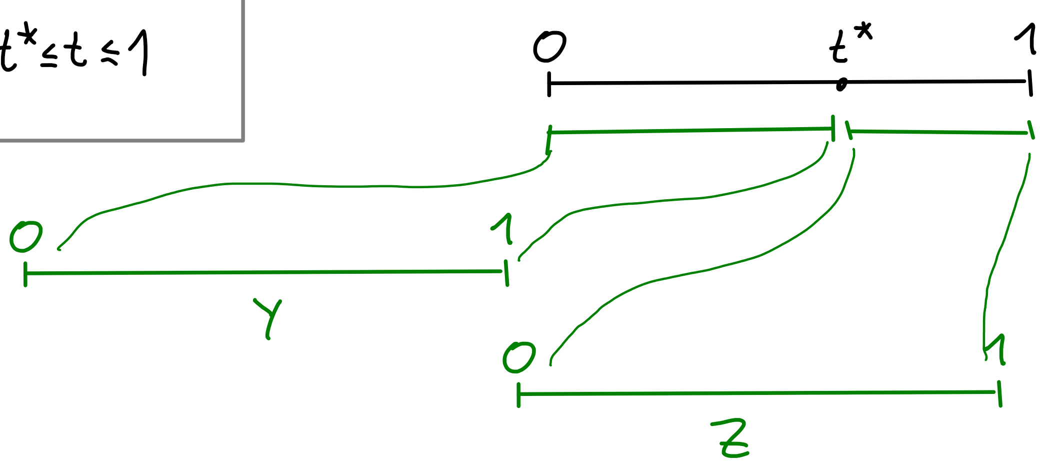
$(i \le k \le j)$

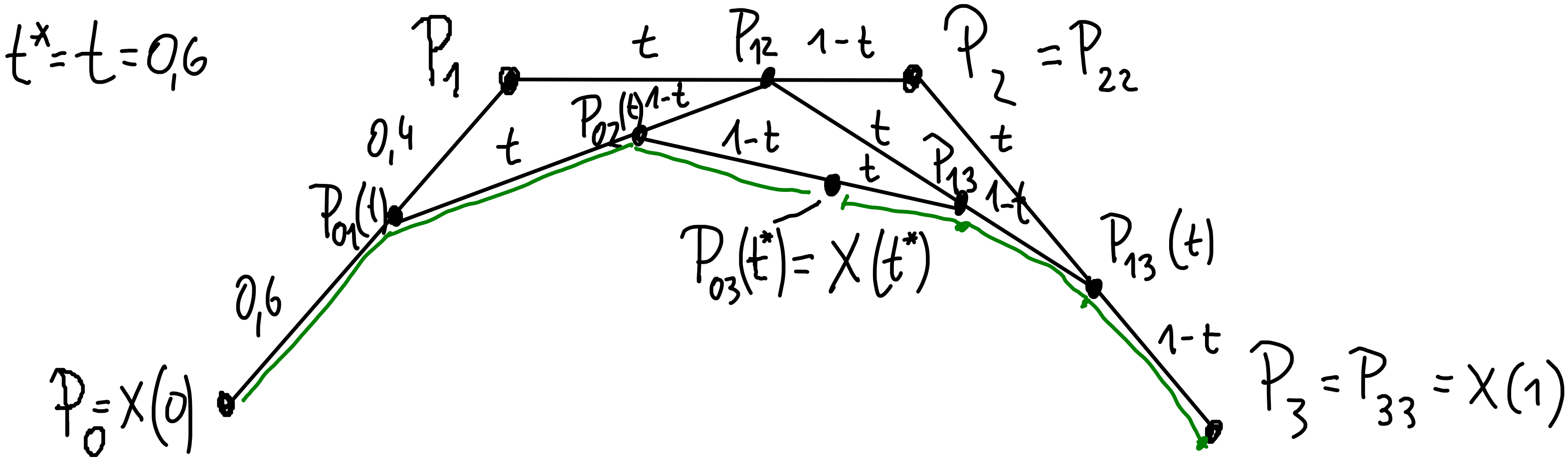
j-i Schritte, davon
k-i ~~j-k~~ nach links und
j-k ~~k-j~~ nach rechts, in beliebiger Reihenfolge. Anzahl = $\binom{j-i}{k-\cancel{j}\,i}$

Unterteilung von Bézier-Splines



$$\boxed{\begin{aligned} X(t) &= Y\left(\frac{t}{t^*}\right), \quad 0 \le t \le t^* \\ X(t) &= Z\left(\frac{t-t^*}{1-t^*}\right), \quad t^* \le t \le 1 \end{aligned}}$$





Der Kurventeil von $X(0)$ bis $X(t^*)$ ist ein Bézierspline mit Kontrollpolygon $P_0, P_{01}(t^*), P_{02}(t^*), \dots, P_{0d}(t^*) = X(t^*)$.

Der Kurventeil von $X(t^*)$ bis $X(1)$ ist ein Bézierspline mit Kontrollpolygon $P_{0d}(t^*) = X(t^*), P_{1d}(t^*), \dots, P_{d-1\,d}(t^*), P_d$.

zu zeigen: (linke Hälfte)

$$\sum_{i=0}^{d}{}' B_i^d(t)\,\underbrace{P_{0i}(t^*)} \overset{!}{=} X(t \cdot t^*) \qquad 0 \le t \le 1$$

$$\underbrace{\sum_{i=0}^{d}{}' \binom{d}{i} t^i (1-t)^{d-i} \sum_{k=0}^{i}{}' \binom{i}{k} t^{*k} (1-t^*)^{i-k} P_k} \overset{!}{=} \sum_{k=0}^{d}{}' \underbrace{\binom{d}{k} (tt^*)^k (1-tt^*)^{d-k}} P_k$$

$$\sum_{0 \le k \le i \le d} \cdots = \sum_{k=0}^{d}{}' \underbrace{\sum_{i=k}^{d}{}' (\dots)} P_k$$

$$\sum_{i=k}^{d}{}' \binom{d}{i} t^i (1-t)^{d-i} \binom{i}{k} \cancel{t^{*k}} (1-t^*)^{i-k} \overset{!}{=} \underbrace{\binom{d}{k} (t\cancel{t^*})^k (1-tt^*)^{d-k}}$$

$$\frac{\cancel{d!}}{\cancel{i!}\,(d-i)!} \; \frac{\cancel{i!}}{\cancel{k!}\,(i-k)!} \qquad \frac{\cancel{d!}}{\cancel{k!}\,(d-k)!}$$

$$\sum_{i=k}^{d} \frac{(d-k)!}{(d-i)!\,(i-k)!}\, t^{i} (1-t)^{d-i} (1-t^{*})^{i-k} \stackrel{!}{=} t^{k} (1-tt^{*})^{d-k}$$

$$\sum_{j=0}^{d-k} \frac{(d-k)!}{(d-k-j)!\, j!}\, \underline{t}^{j+\cancel{k}} (1-t)^{d-k-j} \underline{(1-t^{*})^{j}} \stackrel{!}{=} \cancel{t^{k}} (1-tt^{*})^{d-k}$$

$j \equiv i-k, \quad i \equiv j+k$

$$\sum_{j=0}^{d-k} \binom{d-k}{j} (t-tt^{*})^{j} (1-t)^{d-k-j}$$

$$\left[(\cancel{t}-tt^{*}) + (1-\cancel{t}) \right]^{d-k} \quad \checkmark \qquad \square$$