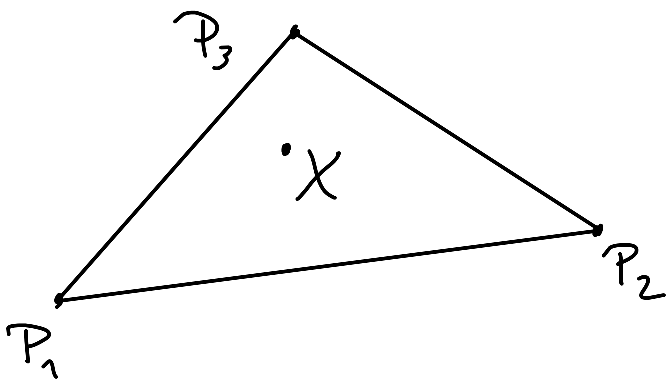
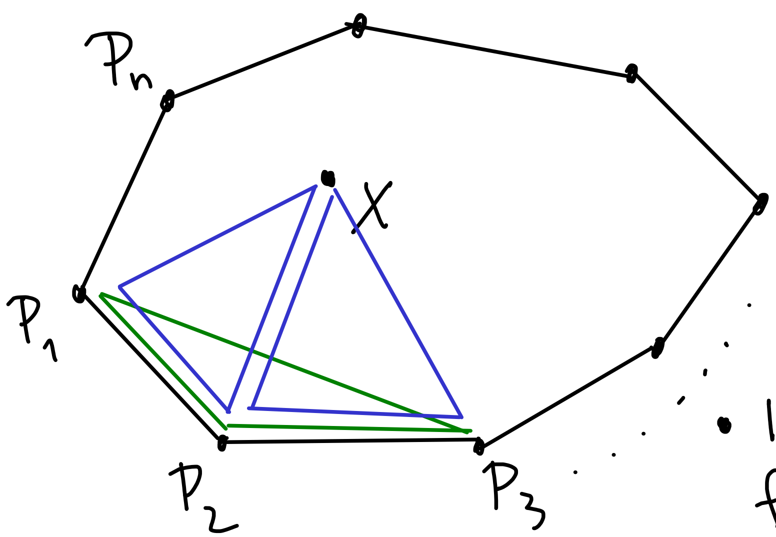
Interpolation auf Polygonen, verallgemeinerte baryzentrische Koordinaten



$$X = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$$
$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$\lambda_1, \lambda_2, \lambda_3 \geq 0$ im Dreieck



$$X = \sum_{i=1}^{n} \lambda_i P_i$$
$$\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$$

Anwendungen:

- Interpolation

$$f(X) := \lambda_1 f(P_1) + \lambda_2 f(P_3) + \dots + \lambda_n f(P_n)$$

interpoliert   gegeben

- Deformation, Morphing

Michael Floater (2014) : Wachspress and mean value coordinates

Wachspress-Koordinaten (1975)

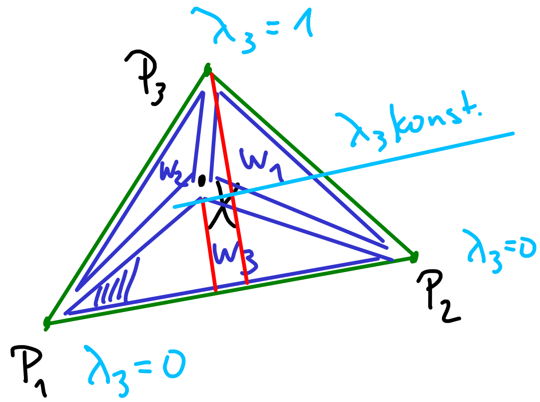
$$w_i := \frac{|P_{i-1} P_i P_{i+1}|}{|P_{i-1} P_i X| \cdot |P_i P_{i+1} X|} \qquad \lambda_i := \frac{w_i}{w_1 + w_2 + \dots + w_n}$$

Ma kann alle w_i mit der gleichen Konstante multiplizieren; das führt auf die gleichen λ_i.

Spezialfall $n=3$

$$w_i := \frac{\cancel{|P_{i-1}P_iP_{i+1}|} \cdot |P_{i+1}P_{i+2}X|}{\cancel{|P_{i-1}P_iX| \cdot |P_iP_{i+1}X| \cdot |P_{i+1}P_{i+2}X|}}$$

$$\lambda_3 = \frac{|P_1P_2X|}{|P_1P_2P_3|}$$



Umschreiben der Interpolationseigenschaft:

$$\sum_{i=1}^{n} \lambda_i X = \sum_{i=1}^{n} \lambda_i P_i \qquad \sum_{i=1}^{n} \lambda_i (P_i - X) = 0$$

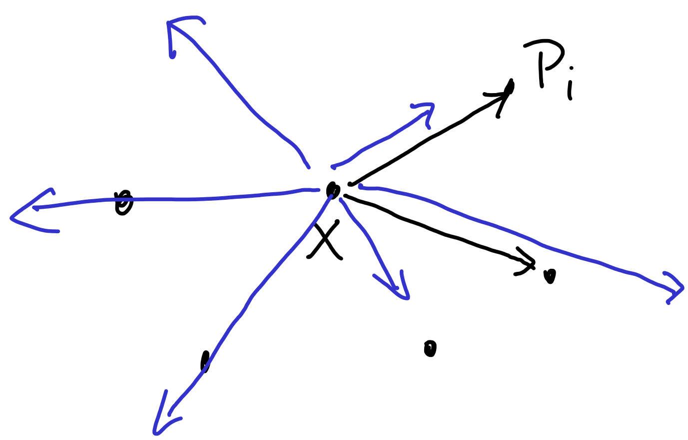
$$\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$$

$$\lambda_i := \frac{w_i}{w_1 + w_2 + \dots + w_n} \qquad \boxed{\sum_{i=1}^{n} w_i (P_i - X) = 0}$$

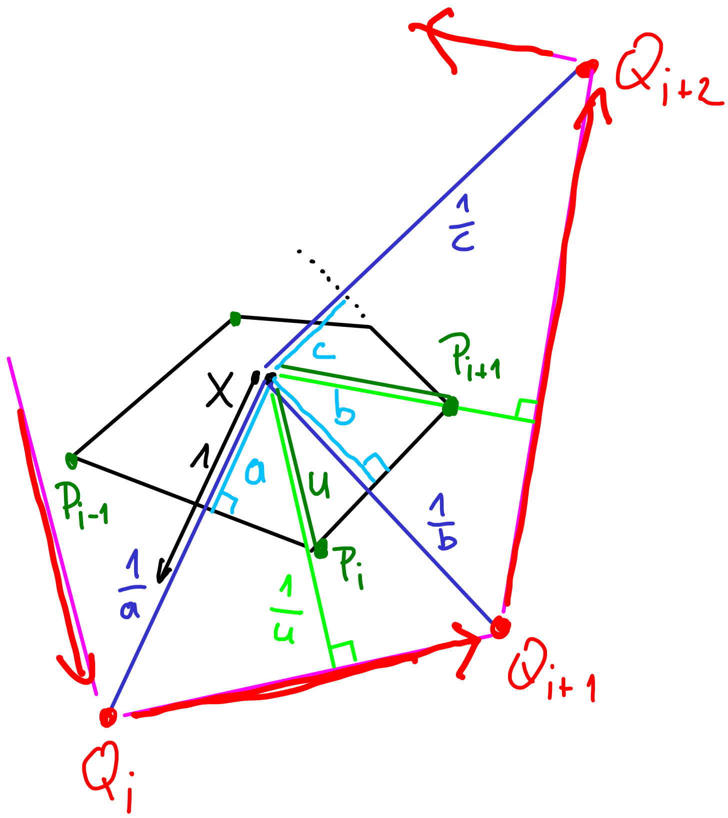
Federkonstante

Kraft

Gleichgewicht



Das polare Polygon



zu zeigen:

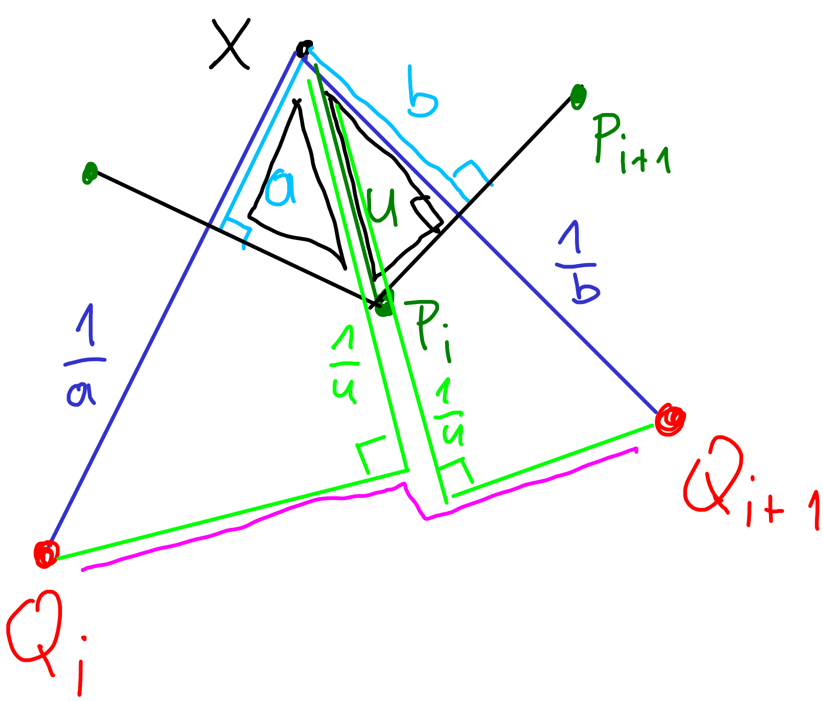
$$\boxed{\sum_{i=1}^{n} w_i (P_i - X) = 0}$$

Behauptung:

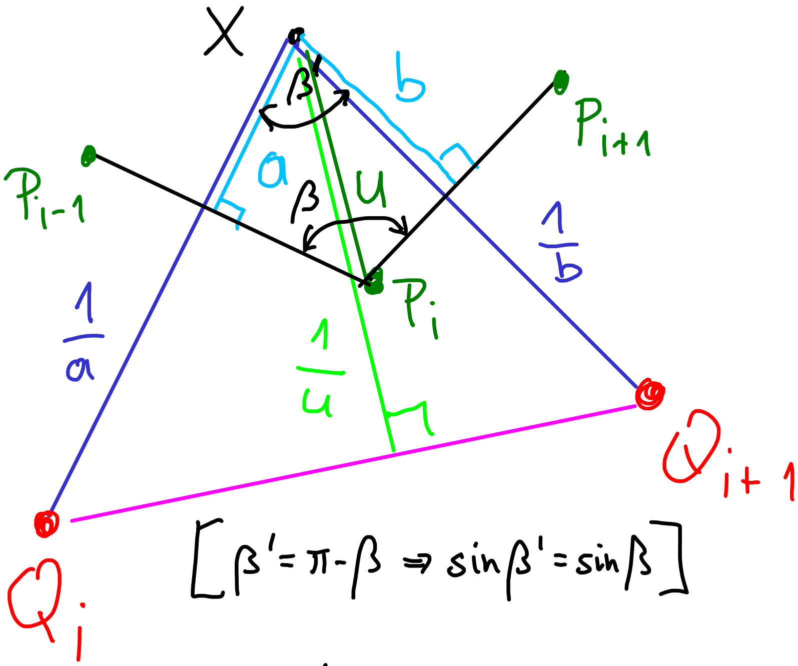
$$w_i (P_i - X)^{\perp} = 2(Q_{i+1} - Q_i)$$

$$\sum_{i=1}^{n} w_i (P_i - X)^{\cancel{\perp}} = 0 \quad \checkmark$$

Das polare Polygon zum polaren ist das ursprüngliche:



$$u : b = \frac{1}{b} : \frac{1}{u}$$



$$\left[\beta' = \pi - \beta \Rightarrow \sin\beta' = \sin\beta\right]$$

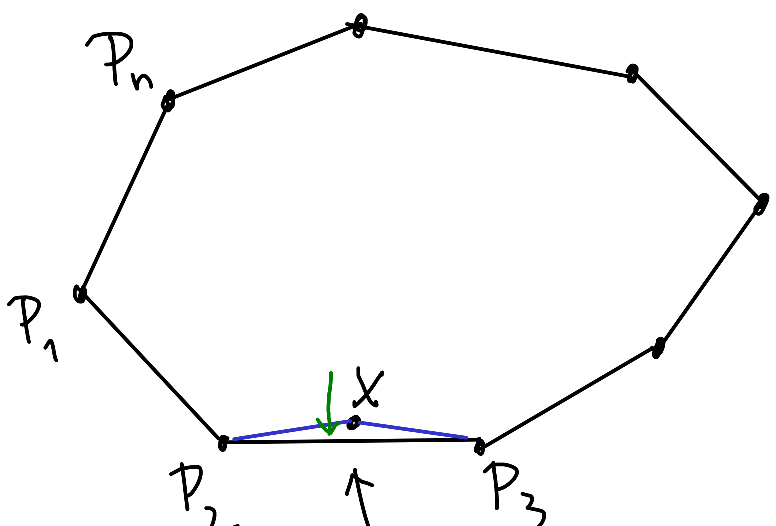
Behauptung:

$$w_i\,(P_i - X)^{\perp} = 2\cdot(Q_{i+1} - Q_i)$$

$$\Leftrightarrow \quad w_i\, u = \overline{Q_i Q_{i+1}}\cdot 2$$

$$w_i = \frac{|P_{i-1}P_iP_{i+1}|}{|P_{i-1}P_iX|\cdot|P_iP_{i+1}X|} = \frac{\cancel{\tfrac12}\cdot\sin\beta\cdot\cancel{\overline{P_{i-1}P_i}}\cdot\cancel{\overline{P_iP_{i+1}}}}{\cancel{\tfrac12}\cdot\cancel{\overline{P_{i-1}P_i}}\cdot a\cdot\tfrac12\,\cancel{\overline{P_iP_{i+1}}}\cdot b}$$

$$= 2\sin\beta\cdot\frac{1}{ab} = 2\sin\beta'\cdot\frac{1}{a}\cdot\frac{1}{b} = 4\,|XQ_iQ_{i+1}| = 4\cdot\tfrac12\,\overline{Q_iQ_{i+1}}\cdot\frac{1}{u} \qquad \square$$



w_2 und w_3 gehen $\to \infty$

$$w_i = \frac{|P_{i-1}P_iP_{i+1}|}{|P_{i-1}P_iX|\cdot|P_iP_{i+1}X|}$$

$w_i \geq 0 \Rightarrow \lambda_i \geq 0$ falls X in $P_1 \dots P_n$ liegt.

Alle Ausdrücke w_i werden mit $|P_2P_3X|$ multipliziert.

(wenn X nahe der Kante P_2P_3 ist)