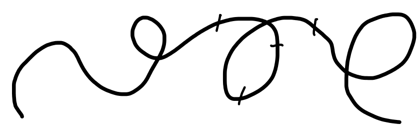
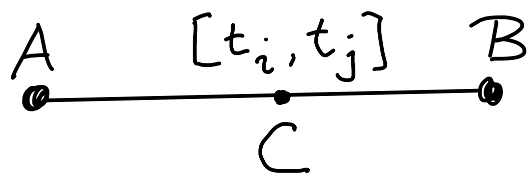
B-Splines, kubisch (Grad $d=3$)



Kontrollpolygon $P_0 P_1 P_2 \ldots P_n$

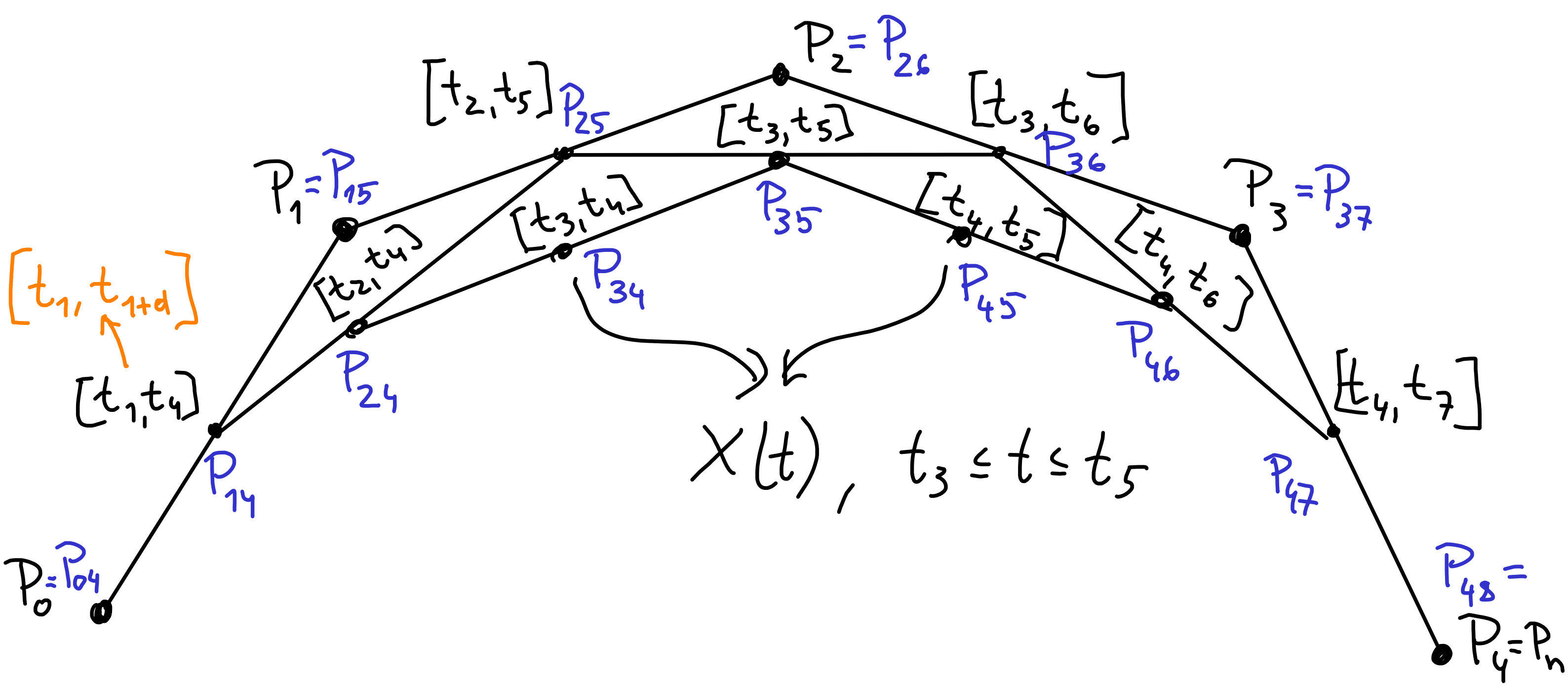
Knotenwerte $[t_0 \le]\, t_1 \le t_2 \le t_3 \le \ldots \le t_{n+d} [+1]$ z.B. $0, 1, 2, 3, \ldots$

Notation:

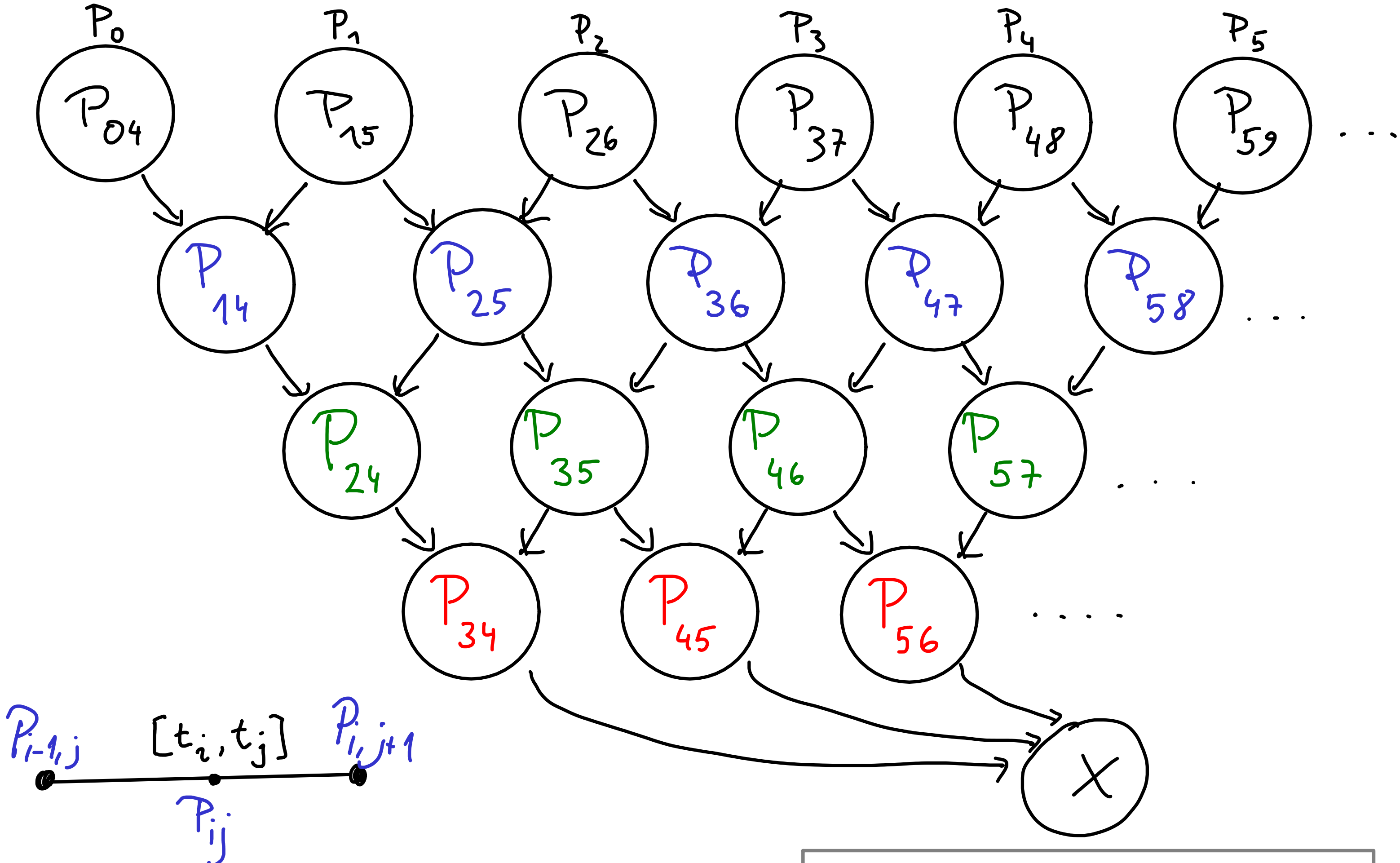


$i<j$

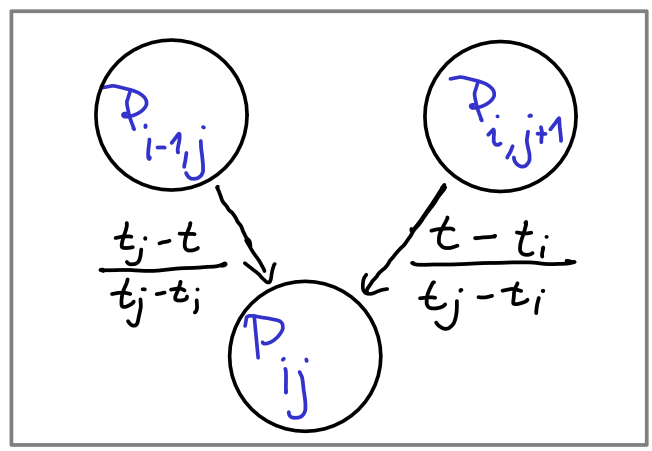
$$C = \frac{t - t_i}{t_j - t_i} B + \frac{t_j - t}{t_j - t_i} A \qquad (t_i \le t \le t_j)$$



$X(t)$ existiert von $t = t_d$ bis $t = t_{n+1}$.



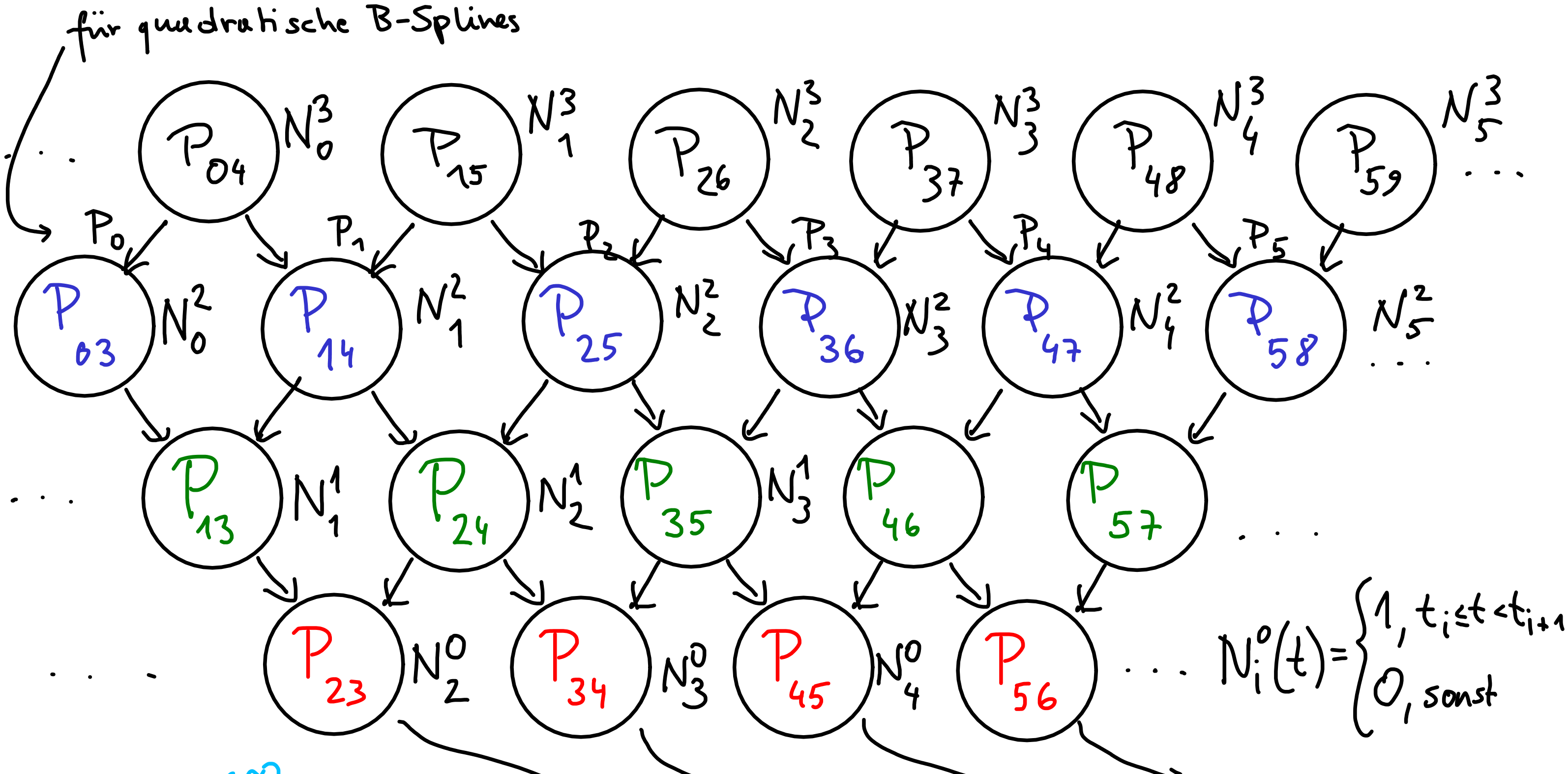
$$P_{ij} = \frac{t - t_i}{t_j - t_i} P_{i,j+1} + \frac{t_j - t}{t_j - t_i} P_{i-1,j}$$



$$X(t) = \sum_{i=0}^{n} \underbrace{N_i^d(t)}_{\text{stückweise Polynomfunktionen (B-spline-Funktionen)}} \cdot P_i \quad (t_d \leq t \leq t_{n+1})$$

LOKALITÄT:

- In jedem Intervall $t_i \leq t \leq t_{i+1}$ hängt $X(t)$ nur von $d+1$ aufeinanderfolgenden Kontrollpunkten ab: $P_{i-d}, P_{i-d+1}, \dots, P_i$.
- Jeder Kontrollpunkt P_i beeinflusst nur $d+1$ aufeinanderfolgende Stücke $t_i \leq t \leq t_{i+d+1}$.



X

$$X(t) = \sum_{i=\cancel{0}-\infty}^{\cancel{n}\,\infty} N_i^d(t) \cdot P_{i,i+d+1}$$

$$X(t) = \sum_{i=\cancel{0}-\infty}^{\cancel{n}\,\infty} N_i^{d-1}(t) \cdot P_{i,i+d} =$$

$$\boxed{P_{ij} = \frac{t-t_i}{t_j-t_i} P_{i,j+1} + \frac{t_j-t}{t_j-t_i} P_{i-1,j}}$$

$$= \sum_{i=\cancel{0}-\infty}^{\cancel{n}-\infty} N_i^{d-1}(t) \cdot \frac{t-t_i}{t_{i+d}-t_i} P_{i,i+d+1} + \sum_{i=\cancel{0}\,\cancel{-1}\,-\infty}^{\cancel{n-1}\,-\infty} N_{i+1}^{d-1}(t) \frac{t_{i+d+1}-t}{t_{i+d+1}-t_{i+1}} P_{\cancel{i-1},\,i+d+1}$$

$$= \sum_{i=\cancel{0}-\infty}^{\cancel{n}-\infty} \underbrace{\left(N_i^{d-1}(t) \cdot \frac{t-t_i}{t_{i+d}-t_i} + N_{i+1}^{d-1}(t) \cdot \frac{t_{i+d+1}-t}{t_{i+d+1}-t_{i+1}} \right)}_{=N_i^d(t)} P_{i,i+d+1}$$

Rekursion : $d \geq 1$

$$\boxed{N_i^d(t) = N_i^{d-1}(t) \cdot \frac{t-t_i}{t_{i+d}-t_i} + N_{i+1}^{d-1}(t) \cdot \frac{t_{i+d+1}-t}{t_{i+d+1}-t_{i+1}}}$$

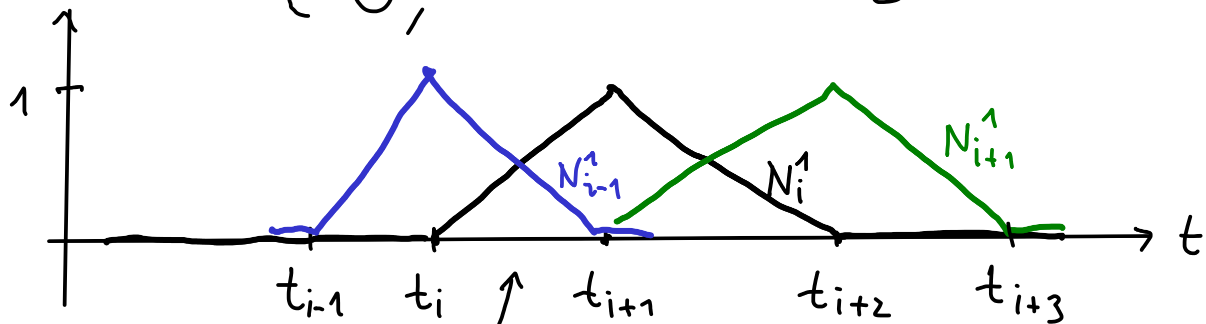
stückweise Funktionen vom Grad d

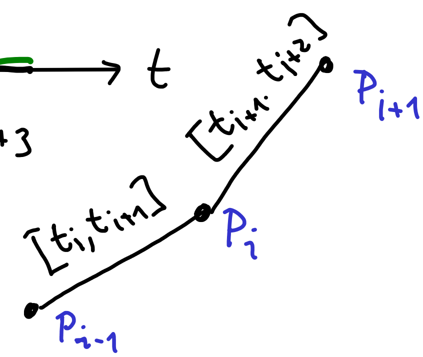
Rekursionsanker $d=0$

$$\boxed{N_i^0(t)=\begin{cases}1, & t_i \le t < t_{i+1}\\ 0, & \text{sonst}\end{cases}}$$

$d=1$

$$N_i^1(t)=\begin{cases}\dfrac{t-t_i}{t_{i+1}-t_i}, & t_i\le t<t_{i+1}\\[2mm] \dfrac{t_{i+2}-t}{t_{i+2}-t_{i+1}}, & t_{i+1}\le t<t_{i+2}\\[2mm] 0, & \text{sonst.}\end{cases}$$





$t_i\le t\le t_{i+1}$ $\quad X(t)=\dfrac{t-t_i}{t_{i+1}-t_i}P_i+\dfrac{t_{i+1}-t}{t_{i+1}-t_i}P_{i-1}$

$$\boxed{\sum_{i=-\infty}^{\infty} N_i^d(t)=1}$$

Gleichmäßige (uniforme) B-Splines : $\quad t_i=i \qquad (t_i=i+k)$ Konstant

$$\boxed{N_i^d(t)=N_i^{d-1}(t)\cdot\frac{t-i}{d}+N_{i+1}^{d-1}(t)\cdot\frac{i+d+1-t}{d}}$$

$$N_i^1(t)=\begin{cases}t-i & i\le t\le i+1\\ i+2-t & i+1\le t\le i+2\end{cases}$$

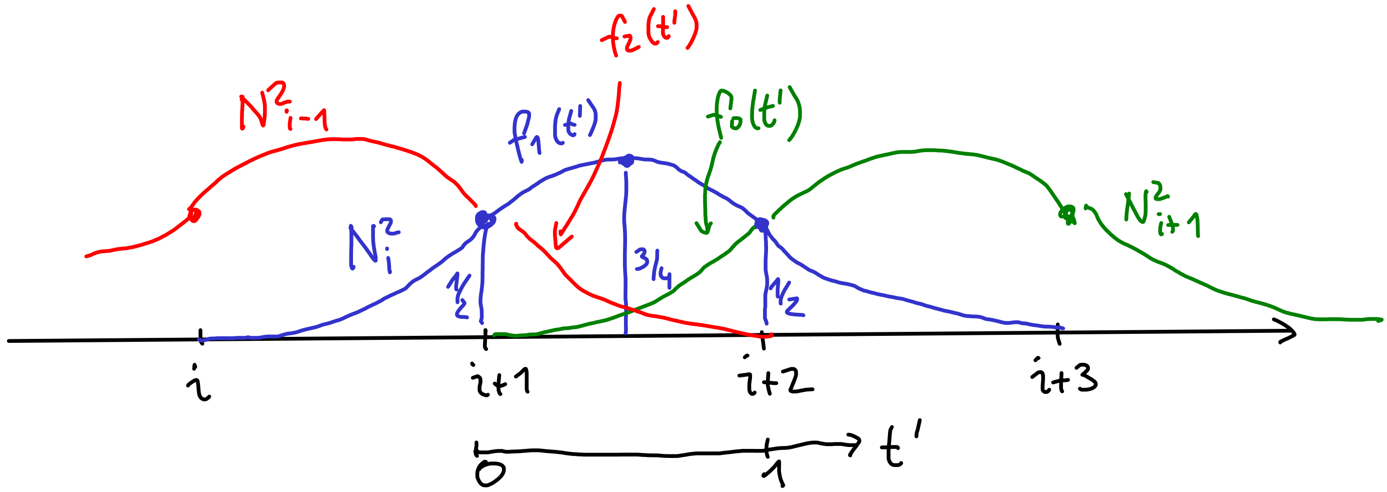
$$N_i^2(t)=\begin{cases}(t-i)\,\dfrac{t-i}{2} & i\le t\le i+1 \quad t'=t-i\\[2mm] (i+2-t)\cdot\dfrac{t-i}{2}+(t-(i+1))\,\dfrac{i+3-t}{2} & i+1\le t\le i+2 \quad t'=t-i-1\\[2mm] (i+3-t)\,\dfrac{i+3-t}{2} & i+2\le t\le i+3 \quad t'=t-i-2\\[2mm] 0, & \text{sonst} \quad 0\le t'\le 1\end{cases}$$

$$N_i^2(t) = \begin{cases} t'^2 \cdot \frac{1}{2} & f_0(t') \\ (1-t')(t'+1)\cdot\frac{1}{2} + t'(2-t')\cdot\frac{1}{2} = \frac{1}{2}\left[1-t'^2+2t'-t'^2\right] & f_1(t') \\ (1-t')^2 \cdot \frac{1}{2} & f_2(t') \end{cases}$$

$$= \frac{1}{2}\left[1+2t'-2t'^2\right]$$

$t'=\frac{1}{2}$: $\frac{3}{4}$

$$= \frac{3}{4} - \left(t'-\frac{1}{2}\right)^2$$



$$N_i^2(t) = \begin{cases} f_0^2(t-i) & i \le t \le i+1 \\ f_1^2(t-i-1) & i+1 \le t \le i+2 \\ f_2^2(t-i-2) & i+2 \le t \le i+3 \\ 0 & \text{sonst.} \end{cases}$$

$$f_0^2(t) + f_1^2(t) + f_2^2(t) \equiv 1$$

$$f_0^2(t) = t^2/2$$
$$f_1^2(t) = \frac{1}{2}\left(-2t^2+2t+1\right)$$
$$f_2^2(t) = \frac{1}{2}(1-t)^2 = \frac{1}{2}\left(t^2-2t+1\right)$$

$$\begin{pmatrix} f_2^2(t) \\ f_1^2(t) \\ f_0^2(t) \end{pmatrix} = \underbrace{\frac{1}{2}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}}_{\text{Basismatrix für quadratische B-splines}} \cdot \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix}$$

- Ordnung = Grad + 1
- unterschiedliche Konventionen / Notationen