Generalized Vandermonde Determinants

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 \\ a_1^4 & a_2^4 & a_3^4 & a_4^4 & a_5^4 \end{vmatrix}$$

$$\begin{aligned} &= (a_2-a_1)(a_3-a_1)(a_4-a_1)(a_5-a_1) \\ &\quad \times (a_3-a_2)(a_4-a_2)(a_5-a_2) \\ &\quad \times (a_4-a_3)(a_5-a_3) \\ &\quad \times (a_5-a_4) \quad = V = V(a_1, \dots, a_n) \end{aligned}$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 \\ a_1^5 & a_2^5 & a_3^5 & a_4^5 & a_5^5 \end{vmatrix} = \sum^{n!} \left(\pm\, a_1^3 a_2^0 a_3^1 a_4^5 a_5^2 \right)$$

$$\begin{aligned} &= (a_2-a_1)(a_3-a_1)(a_4-a_1)(a_5-a_1) \\ &\quad \times (a_3-a_2)(a_4-a_2)(a_5-a_2) \\ &\quad \times (a_4-a_3)(a_5-a_3) \\ &\quad \times (a_5-a_4) \times [a_1+a_2+a_3+a_4+a_5] \end{aligned}$$

$$= V_1$$

[E.R. Heineman, Transactions of the Amer. Math. Soc. 1929]

Vandermonde determinant of these variables. Let us now consider the quotient of any generalized Vandermonde determinant by its Vandermonde determinant. Since both change sign under a transposition, their quotient will remain unchanged, putting it into the class of symmetric functions. A general formula for finding this symmetric function has been the goal of much research in the last half century.

Thus far two ways of treating this subject have been introduced. One of these involves the determination of a general method for finding the quotient of a generalized Vandermonde determinant by the Vandermonde determinant of its variables in terms of symmetric functions. This method

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* Presented to the Society, September 9, 1927; received by the editors in February, 1929.

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = 0$$

roots $a_1, a_2, \ldots, a_n$

$$a_i^n + p_1 a_i^{n-1} + p_2 a_i^{n-2} + \cdots + p_{n-1} a_i + p_n = 0 \quad (i=1,\ldots,n)$$

n equations in n unknowns $p_1, p_2, p_3, \ldots, p_n$.

$$p_1 = \frac{\begin{vmatrix} -a_1^n & a_1^{n-2} & \cdots & a_1 & 1 \\ -a_2^n & a_2^{n-2} & \cdots & a_2 & 1 \\ \vdots & & & & \\ -a_n^n & a_n^{n-2} & \cdots & a_n & 1 \end{vmatrix}}{\begin{vmatrix} a_1^{n-1} & a_1^{n-2} & \cdots & a_1 & 1 \\ a_2^{n-1} & a_2^{n-2} & \cdots & a_2 & 1 \\ \vdots & & & & \\ a_n^{n-1} & a_n^{n-2} & \cdots & a_n & 1 \end{vmatrix}} = \frac{-V_1}{V}$$

[Cramer's rule]

[Vieta's formula]

$$(x-a_1)(x-a_2)\cdots(x-a_n)$$

$$= x^n - \underbrace{(a_1+a_2+\cdots+a_n)}_{-p_1} x^{n-1} + (a_1a_2 + a_1a_3 + \cdots) x^{n-2} - \cdots$$

$$\cdots \pm a_1 a_2 a_3 \cdots a_n$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 \\ a_1^{\bullet} & a_2^{\bullet} & a_3^{\bullet} & a_4^{\bullet} & a_5^{\bullet} \end{vmatrix} \quad \bullet = n-1+k$$

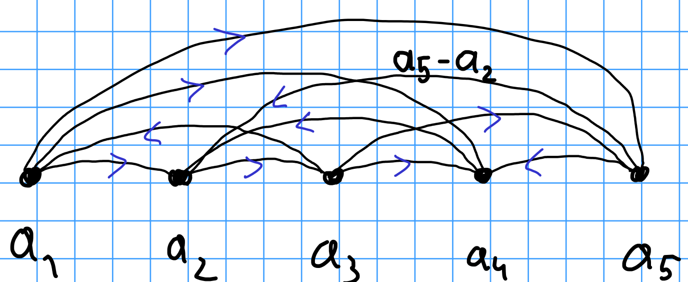
$$= \sum^{n!} \left(\pm a_1^3 a_2^0 a_3^1 a_4^{\bullet} a_5^2 \right) \qquad A$$

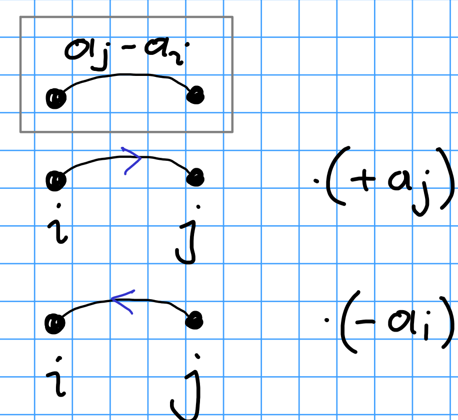
$$= (a_2-a_1)(a_3-a_1)(a_4-a_1)(a_5-a_1) \times (a_3-a_2)(a_4-a_2)(a_5-a_2) \times (a_4-a_3)(a_5-a_3) \times (a_5-a_4) \qquad B$$

$$\times \sum \text{all monomials of degree } k$$

H_k the complete symmetric functions

Example: $H_3 = \sum'_{1 \le i \le j \le k \le n} a_i a_j a_k = a_1^3 + \cdots + a_1^2 a_2 + \cdots + a_1 a_2 a_3 + \cdots$

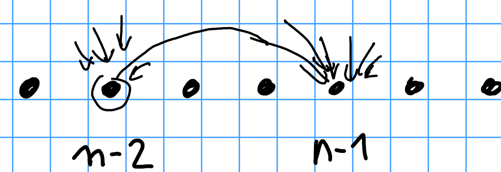




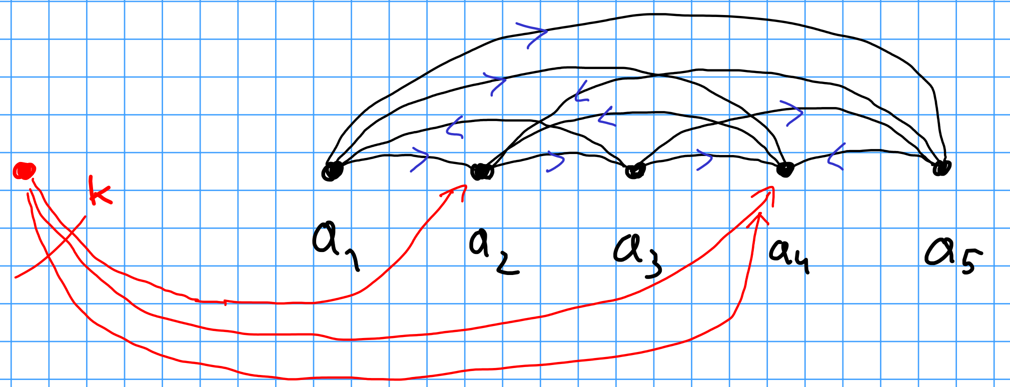
$$B = \sum_{\text{all tournaments}} \prod_i a_i^{\text{indegree}} (-1)^{\#(\text{right-to-left arcs})}$$

$\{\text{indegrees}\} = \{0, 1, 2, \ldots, n-1\}$ .... unique transitive tournament $= A$ (check sign!)

$\sum' \text{indegrees} = \binom{n}{2}$

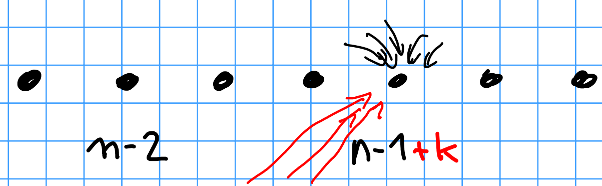


$$\times\left[\sum \text{all monomials of degree } k\right]$$



$$B = \sum_{\substack{\text{all tournaments}\\ +k \text{ extra incoming arcs}}} \prod_i a_i^{\text{indegree}} \, (-1)^{\#(\text{right-to-left arcs})}$$

$\{\text{indegrees}\} = \{0, 1, 2, \dots, n-2, n-1+k\}$ .... UNIQUE ... $= A$



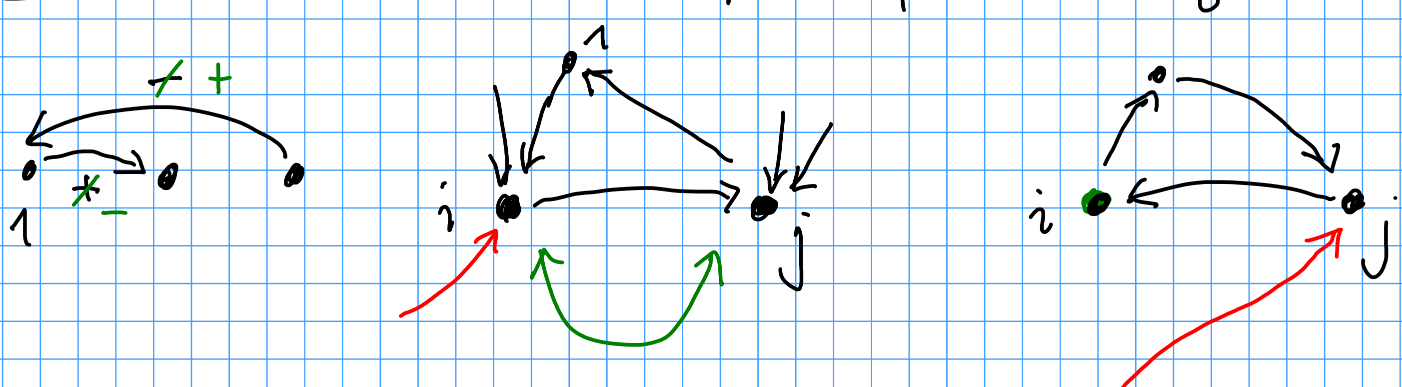
BAD degree sequences

$\boxed{1.}$ duplications

$\boxed{2.}$ the highest degree is not $n-1+k$

} define a sign-changing involution!

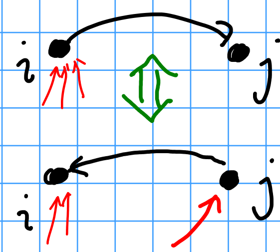
$\boxed{1.}$ Take the lex-smallest duplicate pair $d_i = d_j$



$\boxed{2.}$ distinct degrees, but the highest indegree d_i is $< n-1+k$

- $d_i \geq n$. $\Rightarrow$ i has always at least one red arc

- Find the first "unusual" node j:
  - $i \to j$, or
  - $j \to i$, and j has a red arc



If j does not exist,

outdeg $= 0$, indegree $= n-1$

Heineman 1929, Theorem II, reformulated:

$$\frac{V^{k}}{V} = \underbrace{\left|\begin{matrix} E_1 & E_2 & E_3 & \cdots & E_k \\ 1 & E_1 & E_2 & \cdots & E_{k-1} \\ & 1 & E_1 & \cdots & \\ & & 1 & \ddots & \vdots \\ & 0 & & \ddots & \\ & & & 1 & E_1 \end{matrix}\right|}_{k} \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\end{matrix}}\right\} k$$

E_j = elementary symmetric function of degree j $= \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} a_{i_1} a_{i_2} \cdots a_{i_j}$

Example $k=2$: $\begin{vmatrix} E_1 & E_2 \\ 1 & E_1 \end{vmatrix} = (a_1 + \cdots + a_n)^2 - (a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n)$
$= H_2$

Older literature

* American Journal of Mathematics, vol. 7, pp. 345–352, 380–388; Quarterly Journal of Mathematics, vol. 21, pp. 217–224.

† Proceedings of the Royal Society of Edinburgh, vol. 14, pp. 433–445.

‡ Proceedings of the Royal Society of Edinburgh, vol. 22, pp. 539–542.

§ American Journal of Mathematics, vol. 25, pp. 97–106.

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 \\ a_1^4 & a_2^4 & a_3^4 & a_4^4 & a_5^4 \end{vmatrix} \Rightarrow \begin{vmatrix} a_1^{\lambda_4} & a_2^{\lambda_4} & a_3^{\lambda_4} & a_4^{\lambda_4} & a_5^{\lambda_4} \\ a_1^{1+\lambda_3} & a_2^{1+\lambda_3} & a_3^{1+\lambda_3} & a_4^{1+\lambda_3} & a_5^{1+\lambda_3} \\ a_1^{2+\lambda_2} & a_2^{2+\lambda_2} & a_3^{2+\lambda_2} & a_4^{2+\lambda_2} & a_5^{2+\lambda_2} \\ a_1^{3+\lambda_1} & a_2^{3+\lambda_1} & a_3^{3+\lambda_1} & a_4^{3+\lambda_1} & a_5^{3+\lambda_1} \\ a_1^{4+\lambda_0} & a_2^{4+\lambda_0} & a_3^{4+\lambda_0} & a_4^{4+\lambda_0} & a_5^{4+\lambda_0} \end{vmatrix} \quad (\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_5)$$

$$\begin{aligned} = (a_2-a_1)(a_3-a_1)(a_4-a_1)(a_5-a_1) & \\ \times (a_3-a_2)(a_4-a_2)(a_5-a_2) & \\ \times (a_4-a_3)(a_5-a_3) & \\ \times (a_5-a_4) & \times \underbrace{[\text{some polynomial of degree } \lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4]}_{\text{Schur polynomials}} \end{aligned}$$

symmetric

- the Jacobi-Trudi identities

C.G.J. Jacobi:
De functionibus alternantibus earumque divisione per productum e differentiis elementorum conflatum.
Journal für die reine und angewandte Mathematik 22 (1841), 360–371.

N. Trudi:
Intorno un determinante più generale di quello che suol dirsi determinante delle radici
di una equazione, ed alle funzioni simmetriche complete di queste radici
Rend. Accad. Sci. Fis. Mat. Napoli 3 (1864), 121–134.

A. L. Cauchy: Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs egales et de
signes contraires par suite des transpositions opérés entre les variables qu'elles renferment,
J. Ecole Polyt. 10 (1815), 29–112. ($k=1$ and $k=n$)

Ira Gessel: Tournaments and Vandermonde's determinant
Journal of Graph Theory 3 (1979), 305–307.

Ian P. Goulden: Directed graphs and the Jacobi-Trudi identity
Canad. J. Math. 37 (1985), 1201–1210.

Richard P. Stanley. Enumerative Combinatorics, Vol. 2, 1999, Chap. 7: Symmetric Functions

I.G. MacDonald. Symmetric Functions and Hall Polynomials. Oxford Science Pub., 2nd ed., 1995.

Issai Schur. Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen.
PhD thesis, Berlin, 1901.

[https://en.wikipedia.org/wiki/Lindström–Gessel–Viennot_lemma#Schur_polynomials]