Partitioning a Sequence into Few Monotone Subsequences

Bar Yehuda and Fogel in Acta Inform. '98

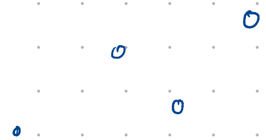
asc/desc

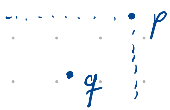
Erdős and Szekeres from '35: Every sequence of n numbers
- has a monotone subsequence of size $\lceil \sqrt{n} \rceil$
- can be partitioned into at most $2\lceil \sqrt{n} \rceil$ monotone subsequences.

$\rightarrow O(n \log n)$

- Assume wlog all numbers are unique.

Embed points into the plane

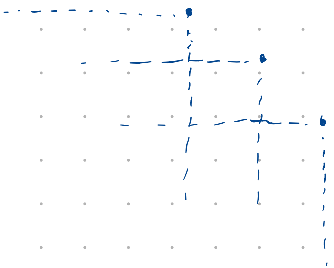
1 3 2 4 $\longrightarrow$ 

- p dominates q 

monotone increasing subsequence ~ points dominate all previous points

- p maximal: There is no point which dominates p.

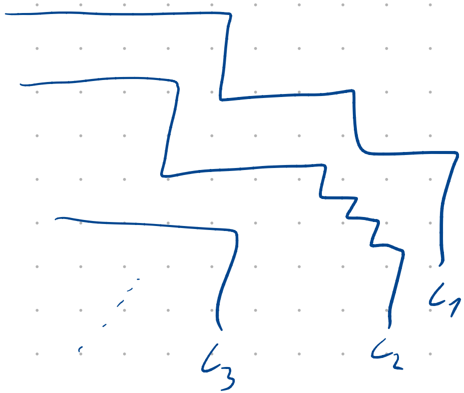
maximal points $M(P)$



- The boundary forms a x and y monotone path
- All other points $P \setminus M(P)$ lie below this path.

Repeating gives us layers

$$L_i(P) = M(P \setminus \bigcup_{j<i} L_j)$$



Observations: • $p \in L_i$, $i > 1$ is dominated

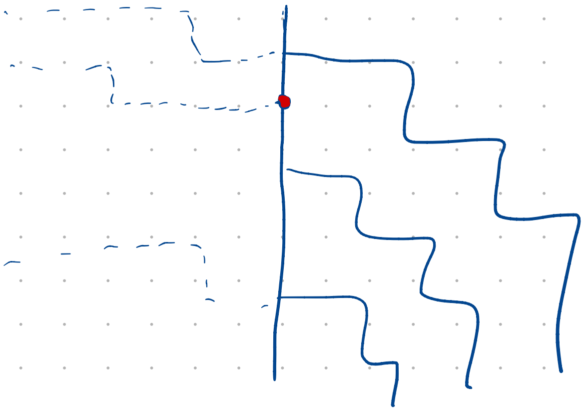
• A point does not dominate any point on the same or higher layer

• This represents all asc monotone subsequences.

↳ In $O(n)$ time we can extract the longest asc monotone subsequence.

Constructing $\mathcal{L}(P) = \{L_1, \dots\}$

- sweep from right to left



$\rightsquigarrow$ $O(n \log n)$ time

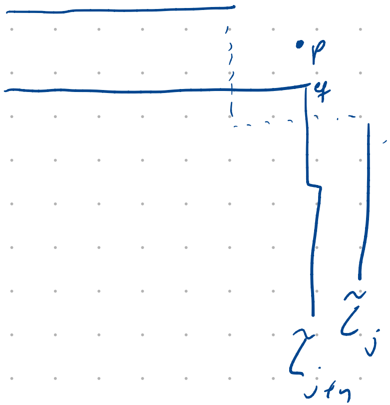
Find the $2\lfloor\sqrt{n}\rfloor$ subsequences:

- Repeat the construction of asc / desc layers + extract $O(n \log n)$

$\rightarrow O(n^{1,5} \log n)$

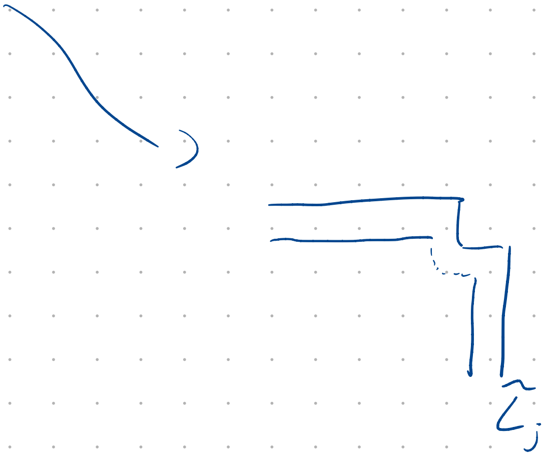
$\hookrightarrow O(n^{1.5})$

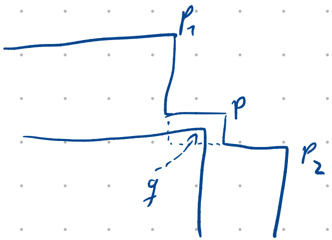
Fix the structure after we extract $D \subseteq P$



- p was deleted
- q witness

} hole





Algorithm 1:

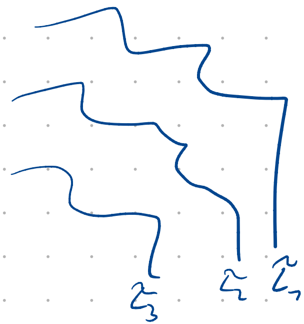
For $i = 1 \ldots \ell$:

For $p \in L_i$ and is a witness:

Find the highest layer $\tilde{L}_j$ where p is a witness for.

Move p to $\tilde{L}_j$

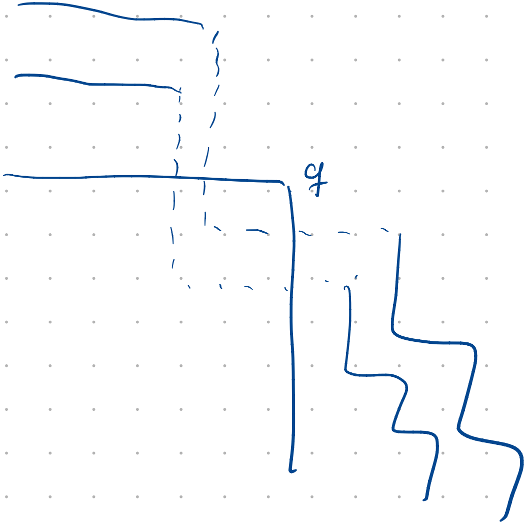
Delete all points $L_i \cap D$ from all $\tilde{L}_j$

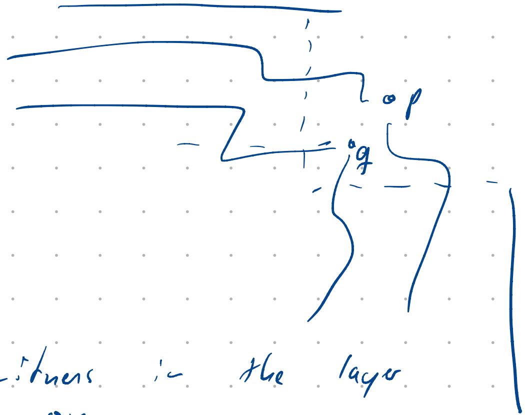


Observations:

- After moving a point: topmost layer or dominated.
- Each point is only witness for one hole per layer
- When q is a witness for layer $\tilde{L}_j$, then it is also a witness for $\tilde{L}_{j+1}$

- when $p \in L_{i+1}$ dominates $q \in L_i$ and q is a witness then p is as well





- All holes have a witness in the layer we're currently working on.

Two types of holes:
- point was deleted
- point is moved

Algorithm 2:

$H^0 \leftarrow \emptyset$

For $i = 1 \dots \ell$:

For $p \in L_i$ and is a witness:

Find the highest layer $\tilde{L}_j$ where p is a witness for.

Move p to $\tilde{L}_j$

$O(L_i + |H^{i-1}|)$

$|H^i| \leq |D|$

$H^i \leftarrow \emptyset$

For $h \in H^{i-1}$:

If there is an outer hole h' candidate:

$H^i \leftarrow H^i \cup \{h'\}$

hole through moving

Delete all points $L_i \cap D$ from all $\tilde{L}_j$

hole through deletion

$\Rightarrow$ Delete a sequence of k numbers in $O(n + k^2)$ time