A Fast Minimum Degree Algorithm and Matching Lower Bound

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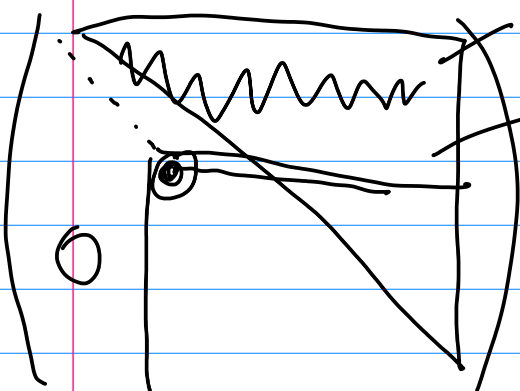
$$\begin{pmatrix} 4 & 0 & 0 & 7 & 3 & \cdots \\ 0 & 1 & 7 & 2 & 6 & \cdots \\ 0 & 2 & 4 & 6 & 0 & \cdots \\ 0\,5 & 3 & 0 & \cancel{0} & \cancel{8} & \cdots \\ 0\,2 & 1 & 0 & \cancel{0} & \cancel{0} & \cdots \\ \vdots \end{pmatrix} \quad -4/5 \quad -1/2$$

SPARSE

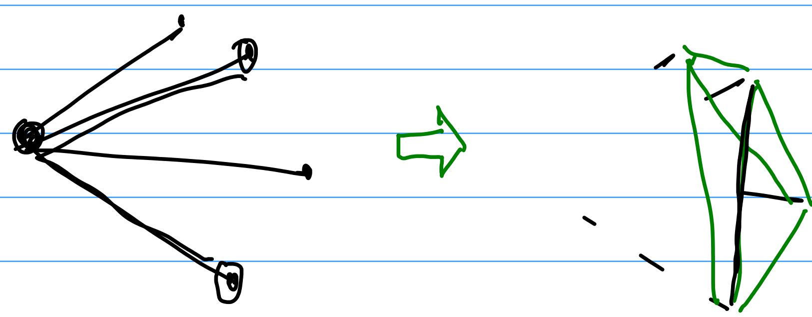
FILL-IN

THIS TALK: SYMMETRIC CASE.



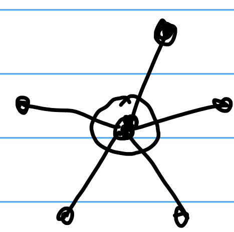
FILL-IN

$m^+ = \#$ non-zeros $= |E^+|$



perf. elim. ordering

chordal graphs.



$O(n^3)$

Adjacency structure

insert / query edges

- adj. matrix $O(1)$
- adj. list $O(\log n)$
- hash $O(1)$

THM: $\boxed{\begin{array}{l}\text{The min.-deg. elimination ordering} \\ \text{a) } O(m\sqrt{m^+}) = O(mn) \\ \text{b) } O(\Delta m^+) \qquad \text{insertions + other operations} \\ \quad \Delta = \text{max. deg in } G_0 \end{array}}$

THIS DOES NOT ACCOUNT FOR THE COST OF ACTUALLY PERFORMING THE ELIMINATION!

$\mathcal{H} \subseteq 2^V \qquad \mathcal{H} = \{C_1, C_2, \dots\}$

edges $\boxed{E = \bigcup_{C \in \mathcal{H}} K_C}$

initially $\mathcal{H} = \{\{i,j\} \mid ij \in E_0\}$

for $i = 1, \dots, n$

v_i = min-deg. vertex

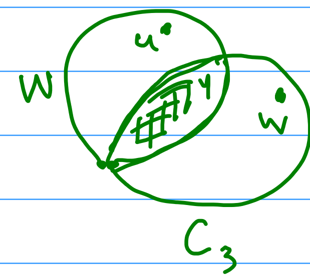
Let $C_1, \dots, C_k$ the clusters $\in \mathcal{H}$ containing v_i

MERGE

replace $C_1, \dots, C_k$ by

$\boxed{(C_1 \cup \dots \cup C_k) \setminus \{v_i\}}$

remove v_i



$W := C_1$

$W := W \cup C_2$

$W := W \cup C_3$

insert edges between $(W - C_3)$ and $(C_3 - W)$

The square of the size of the new cluster is the cost of performing the elimination.

maintenance : $\boxed{L1}$: $O\left(\sum_{C} |C_i|\right)$ ← appear ever

insertions

$\boxed{L2}$: edge u,v is inserted at most $\deg_{G_0}(u)$ times

$\to (m^+ \cdot \Delta)$

$\boxed{L3}$ $\sum'_{uv \in E^+} \min(d(u), d(v)) \le \sqrt{2m^+} \cdot m$

$\to O(\sqrt{m^+} \cdot m)$

$\boxed{L2}$ $g(u,v) := \#\{C \in \mathcal{H} \mid \underline{u \in C},\ v \notin C\}$ $\quad g_0(u,v) \le \deg(u)$

uv added

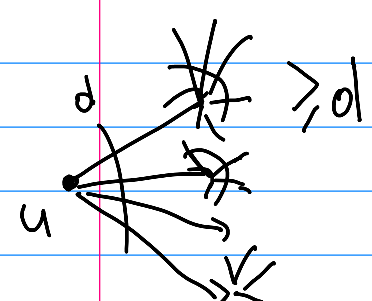
$g \downarrow$

<u>merging</u> cannot increase g (until u or v deleted)

$\boxed{L3}$ $\sum_{uv \in \underline{E^+}} \min(f(u), f(v)) \le \sqrt{2|E^+|} \cdot \sum_{v \in V} f(v)$

↑ orient $\overrightarrow{(u,v)}$ s.t. $\deg_{E^+}(u) \le \deg_{E^+}(v)$

$$\text{L.S.} \le \sum'_{u} \sum'_{\substack{v \\ (u,v) \in \vec{E}}} f(u) \le \sum'_{u} f(u) \underbrace{\deg^+_{\vec{E}}(u)}_{d \,\le\, \sqrt{2|E^+|}}$$



$d \cdot d \le 2|E^+|$