11 Ideas for the 11th Month

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A thematic program on Algebraic Geometry will take place in Berlin during the Winter Semester 2019-20. That special semester is titled **Varieties, Polyhedra, Computation**. One goal is to foster collaborations among different groups from the three Berlin universities.

In the first half of October, a Fall School and an Opening Conference will set the tone. After that, the actual work begins. The present document suggests directions for study and research during the 11th month (November). It offers 11 ideas for open-ended problems.

We start with one that relates to work of Farkas and collaborators on Koszul modules.

Idea 1. *The Grassmannian* $\mathrm{Gr}(1, \mathbb{P}^4)$ *of lines in* 4*-space is a* 6*-dimensional variety in* $\mathbb{P}^9$*. Write its Chow form as the determinant of a* 5×5*-matrix whose entries are linear forms in the* $\binom{10}{6} = 210$ *brackets, ready to be evaluated in a computer algebra system. Do the same with the Hurwitz form, and pass to higher Grassmannians. Determine the Chow polytopes.*

The following is a problem stated in the article *Exponential Varieties* by Michalek et al.

Idea 2. *The Grassmannian* $\mathrm{Gr}(1, \mathbb{P}^n)$ *can be realized as the variety of inverses of* $n \times n$ *Hankel matrices. Which varieties arise from inverting catalecticants of ternary forms? Specifically, determine the variety of inverses for the* 6×6 *catalecticant associated with ternary quartics.*

Geometric Invariant Theory is of considerable current interest in Theoretical Computer Science. The context is the complexity of scaling algorithms. Can we put this into practise?

Idea 3. *Implement a numerical scaling algorithm that tests semistability of projective hypersurfaces. The input is a homogeneous polynomial. The output is either destabilizing one-parameter subgroup, or –ideally– an invariant that does not vanish for the hypersurface.*

Rationality questions are fascinating. The following was suggested by Ritvik Ramkumar.

Idea 4. *Can we find five plane conics such that all* 3264 *tangent conics are defined over* $\mathbb{Q}$*?*

The following idea is inspired by a discussion with Boris Shapiro about complex geometry.

Idea 5. *A complex affine variety of dimension* d *in* $\mathbb{C}^n$ *is a real affine variety of dimension* $2d$ *in* $\mathbb{R}^{2n}$*. Complexifying the latter, we get a complex affine variety of dimension* $2d$ *in* $\mathbb{C}^{2n}$*. What can be said about this process? Even the case* $d = 0$ *and* $n = 1$ *seems quite interesting.*

Let's now come to a theme that is central to algebraic geometry, namely moduli spaces.

Idea 6. *There is a natural embedding of $\overline{\mathcal{M}}_{0,n}$ into $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$. Monin and Rana (2018) offered a conjecture for the ideal generators of this embedding. Can we prove this?*

The moduli space $\mathcal{M}_{23}$ is popular these days as the demarkation point between uniruled and general type. To explore curves of genus 23, one might start with nodal curves of degree $(5,7)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. But how to deal with the general curve? Numerical methods might help.

Idea 7. *Study canonical curves of genus* 23 *using tools from numerical algebraic geometry.*

We now come to a topic in real algebraic geometry, developed by Blekherman et al. (2013), namely the extreme rays of the cone of nonnegative polynomials that are not sums of squares. For ternary sextics, we get the Severi variety of rational sextics. This has codimension 10 in $\mathbb{P}^{27}$, and its degree is the Gromov-Witten number $26,312,976$. The other Hilbert case, namely quartics in four variables, leads to an interesting locus in the space of K3 surfaces.

Idea 8. *The variety of quartic symmetroids in $\mathbb{P}^3$ has codimension* 10 *in the $\mathbb{P}^{34}$ of quartics. Determine its degree. What is the polynomial of lowest degree that vanishes on this locus?*

Here is another nice application that arises in the theory of convex optimization.

Idea 9. *Pataki considered the set of semidefinite programming instances for which strong duality fails. Let's study the corresponding algebraic variety. Dimension? Components?*

Thomas Krämer studies holonomic D-modules on abelian varieties. How to make this concrete? To me, a holonomic D-module is a system of linear partial differential equations with polynomial coefficients whose solution space is finite-dimensional. Represented in `Macaulay2` or `Plural` by left ideals in a Weyl algebra, these can be very useful for statistics.

Idea 10. *Further develop computational tools for D-modules and their characteristic cycles. Focus on those that are of interest in algebraic geometry and geometric representation theory.*

Angela Ortega and Andrea Petracci among the organizers for the Opening Conference (October 7-11). Here is a quiz for these two. Of course, everyone is invited to to help them.

Idea 11. *Most varieties of interest to algebraic geometers are not unirational. Associate one non-unirational variety with each of the 14 names of speakers at the Opening Conference. This should be a variety which, in your opinion, might come up in that person's presentation.*

I hope that you find these ideas worthwhile to think about, and that they will stimulate some interesting discussions among MATH+ afficionados in Berlin. Of course, to find even more people to talk to, you are always welcome to hop on that one-hour train down south, and visit us at MPI Leipzig for the day. Scheduled events during the 11th month include *Buildings, Varieties and Algorithms* and *Developments in the Mathematical Sciences 2019.*

In summary, there is no shortage of good research problems in algebraic geometry and its applications, including those that are accessible to beginning graduate students. For instance, there are still many questions about cubic surfaces left to be answered as well.