Freie Universität Berlin

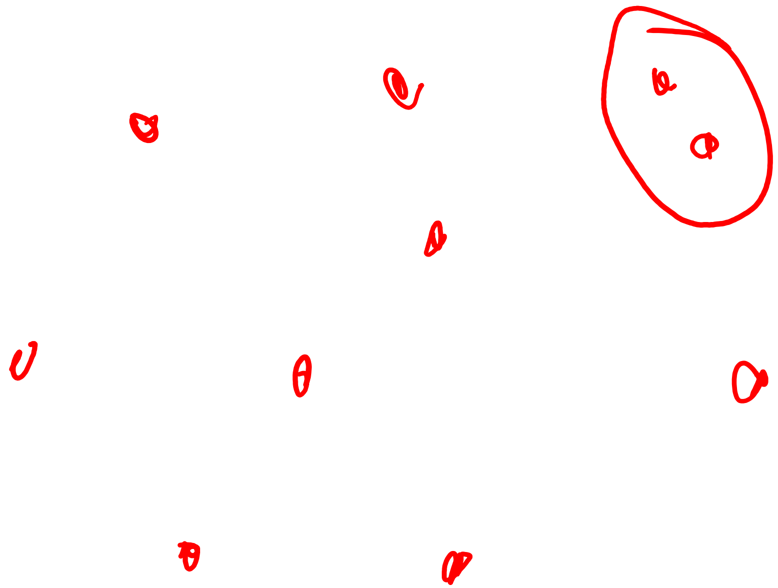
Closest Pairs in Doubling Metrics

Anil Maheshwari — Carleton University
Wolfgang Mulzer — Freie Universität Berlin
Michiel Smid — Carleton University

The Problem

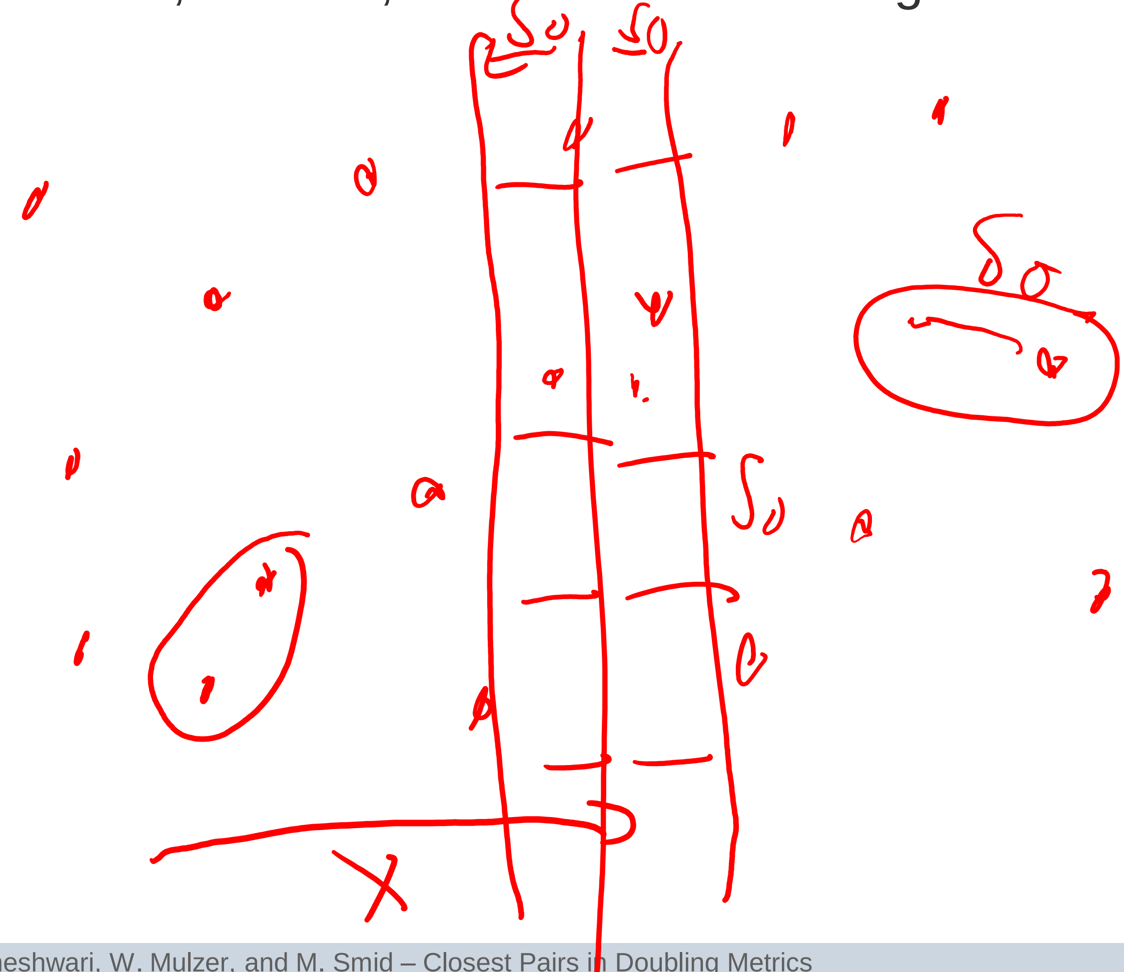
Given: N points P in the plane

Want: closest pair of P



O(N log N) Algorithm by Bentley-Shamos

classic textbook algorithm: sort by x-coordinate, split on median, recurse, combine results using median slab



General Metric Spaces

P: finite set with N elements

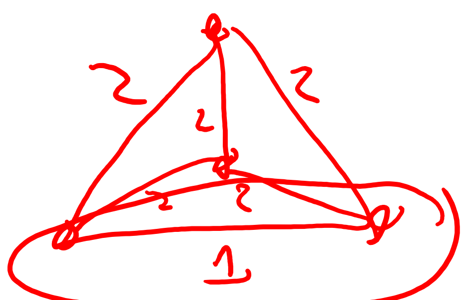
$d: P \times P \to R_{\geq 0}$

1. $d(x, y) = 0 \leftrightarrow x = y$, f.a. $x, y \in P$
2. $d(x, y) = d(y, x)$, f.a. $x, y \in P$
3. $d(x, y) \leq d(x, z) + d(z, y)$, f.a. $x, y, z \in P$

Assumption: Have $O(1)$ oracle to compute $d(x, y)$ for given $x, y \in P$.

Lower bound: Need $\Omega(N^2)$ time in general metric spaces.

Exactly one pairwise distance $= 1$, all other pairwise distances $= 2$.

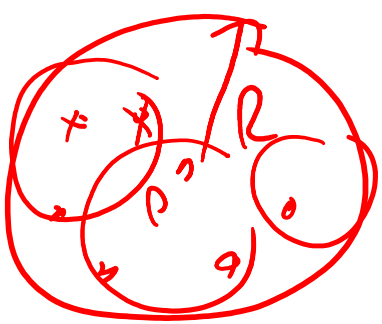


Doubling Dimension

ball $B(p, R) = \{ q \in P \mid d(p, q) \leq R \}$

Suppose every ball $B(p, R)$ in P can be covered by λ balls with radius $R/2$, for every $q \in P$, $R \geq 0$.

doubling dimension of P: $d = \log \lambda$



Doubling Dimension

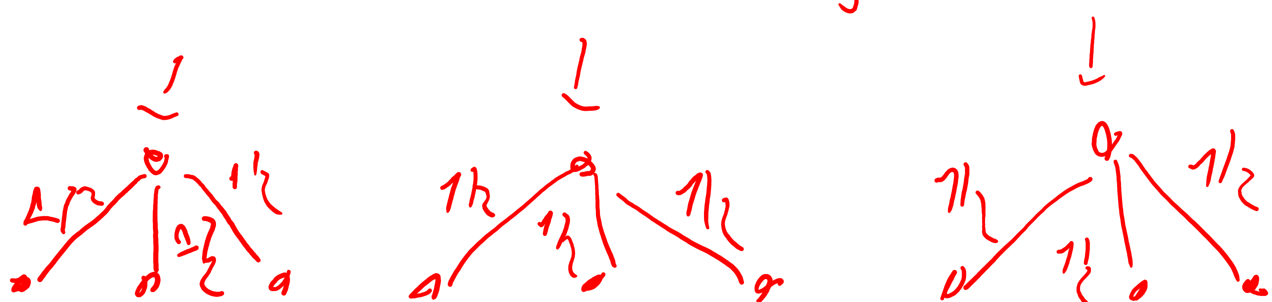
The discrete metric space has doubling dimension log N.

R^2 has doubling dimension log 7.

The doubling dimension in a subspace may go up, but only by a factor of 2.

$\sqrt{N}$

$\log \sqrt{N} = \frac{1}{2} \log N$



$B(p,1) = P$

$B(p, 1/2) = \{p\}$

$\log N$

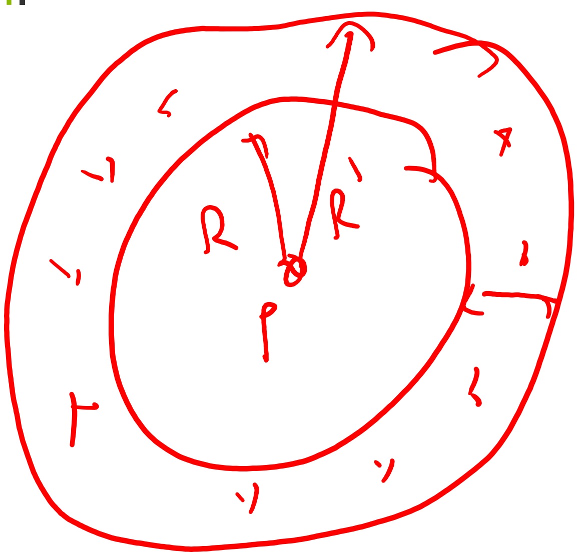
Closest Pairs in Doubling Metrics

doubling metric: metric space with constant doubling dimension

Can still find the closest pair in $O(N \log N)$ expected time.

No coordinates? No Slabs? No Grids?

Use balls and annuli.

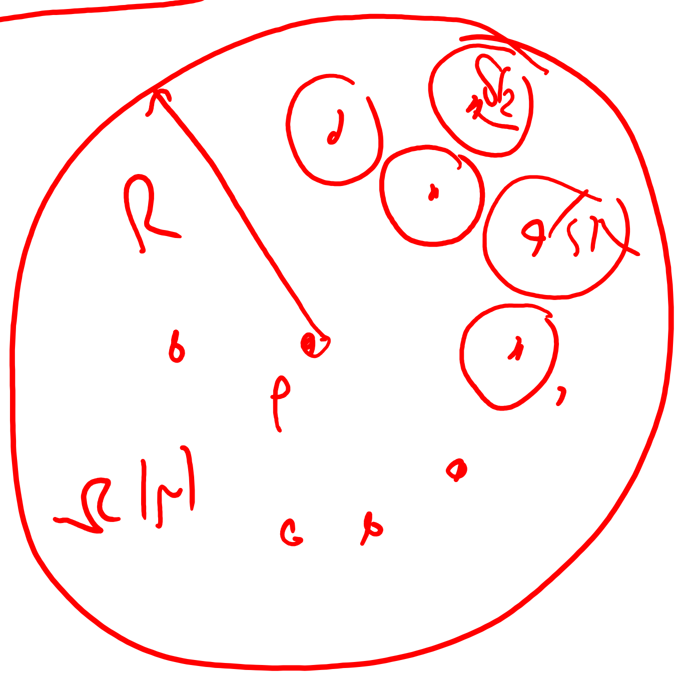


width $= R' - R$

Closest Pairs in Doubling Metrics

First O(N log N) expected time algorithm by Har-Peled and Ali Abam (SSPD) (2010). We make it simpler.

Lemma: Let B be a ball in P with $N/8^d$ points. Then, radius(B) = $\Omega(\delta N^{1/d})$, where δ = closest pair distance.



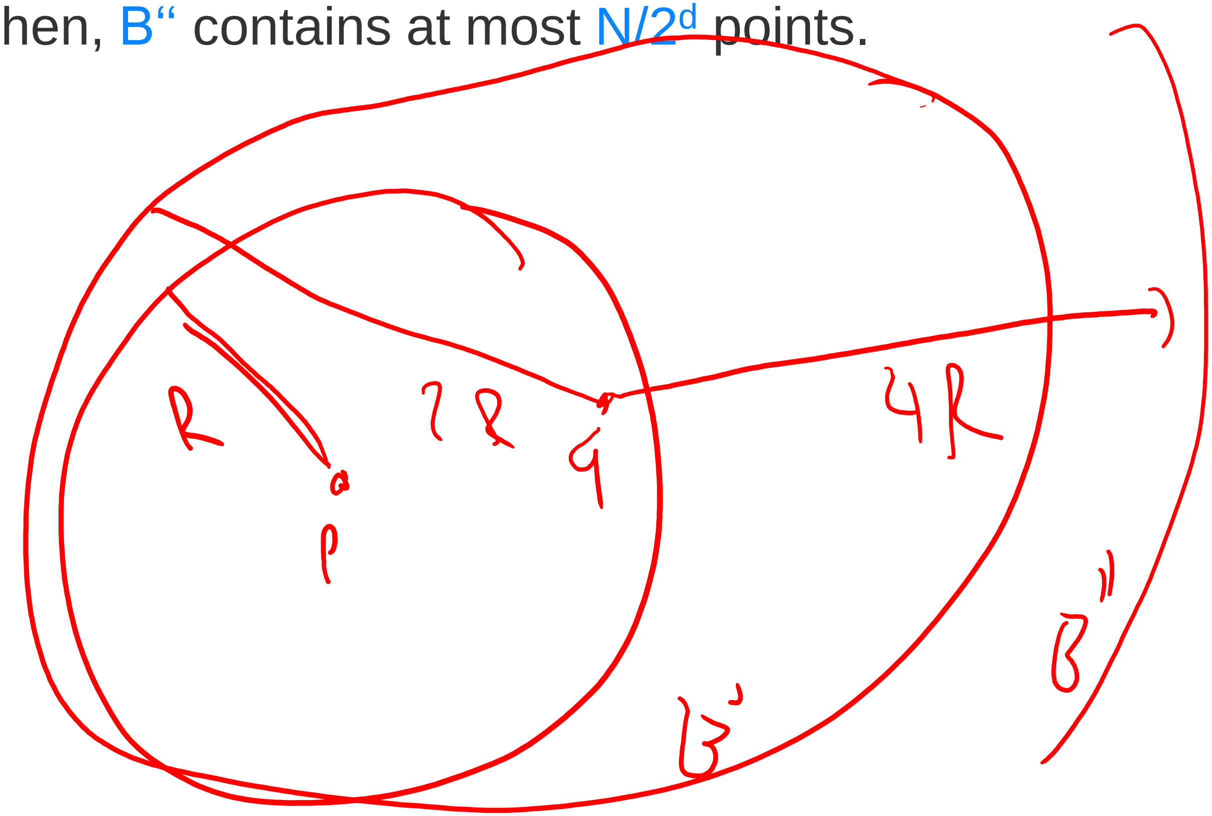
$N \le 2^{d \log \frac{R}{\delta}}$

$N = \left(\frac{R}{\delta}\right)^d$

$R \approx N^{1/d} \cdot \delta$

Closest Pairs in Doubling Metrics

Lemma: Let B be minimum-radius ball in P with $N/8^d$ points. Let $q \in B$ and B‘ minimum-radius ball around q with $N/8^d$ points. Let B“ be concentric ball to B‘ around q with double radius. Then, B“ contains at most $N/2^d$ points.

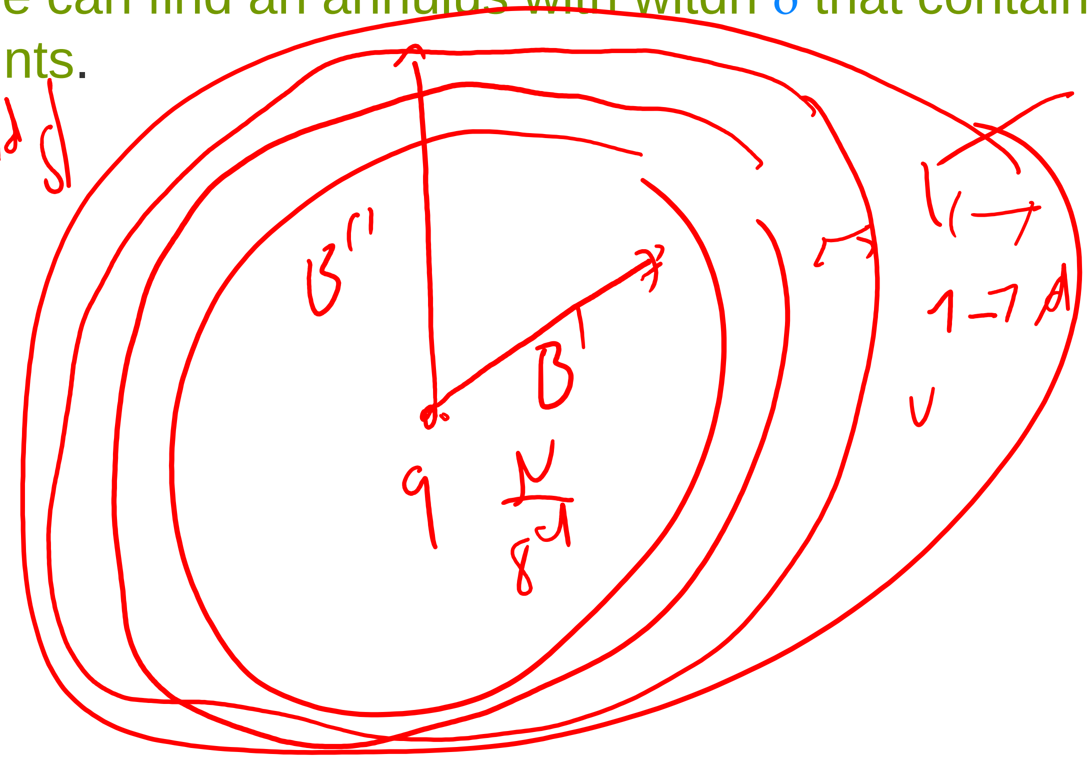


Closest Pairs in Doubling Metrics

Lemma: Let B be minimum-radius ball in P with $N/8^d$ points. Let $q \in B$ and B' minimum-radius ball around q with $N/8^d$ points. Let B'' be concentric ball to B' around q with double radius. Then, B'' contains at most $N/2^d$ points and between B and B'', we can find an annulus with witdh δ that contains $O(N^{1-1/d})$ points.

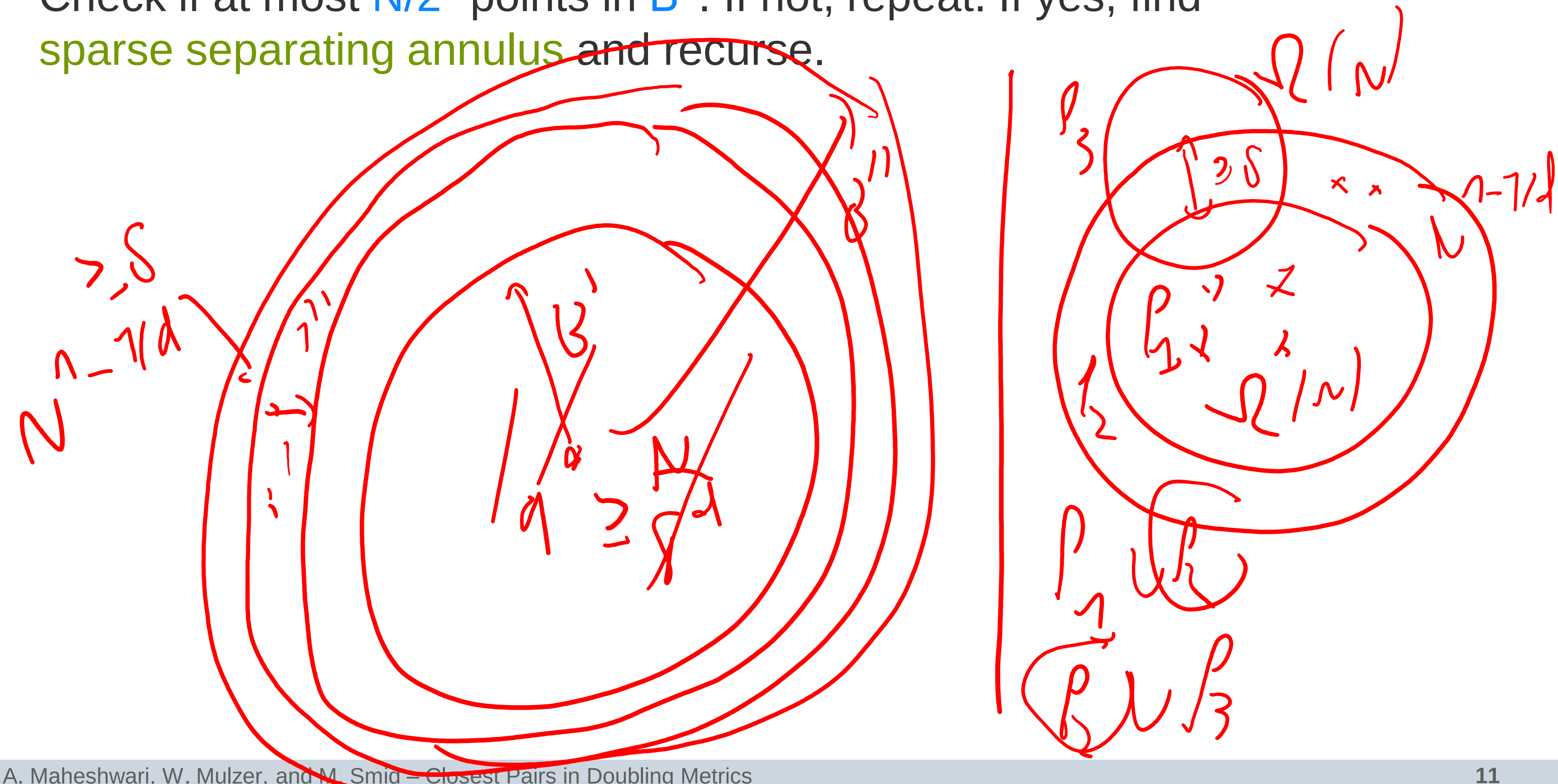
$|B'| = \Omega(N^{1/d}\delta)$

$N^{1/d}$



Closest Pairs in Doubling Metrics

Algorithm: Pick random $q \in P$. With probability $1/8^d$, we have $q \in B$. Take minimum-radius ball B' around q with $N/8^d$ points. Check if at most $N/2^d$ points in B''. If not, repeat. If yes, find sparse separating annulus and recurse.



Closest Pairs in Doubling Metrics

Result: The expected running time of this algorithm is O(N log N).

$$T(N) = T(N_1) + T(N_2) + O(N)$$

$$N_1 + N_2 \leq N + N^{1-1/d}$$

$$N_1, N_2 = \Omega(N)$$