Forbidden submatrices

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Forbidden submatrices

- Only 0-1 matrices in this talk
- Notation: 1s as $\bullet$, 0s omitted:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} \bullet & & \\ & \bullet & \bullet \\ & \bullet & \bullet \end{pmatrix}$$

- No empty rows/columns

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Definitions & simple facts

The *weight* $|M|$ of a 0-1 matrix M is the number of 1-entries in it.

Let $\text{Ex}(n, P)$ be the maximum weight in a $n \times n$ 0-1 matrix avoiding P.

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- $\mathrm{Ex}(n, P) \geq n$ if $P \neq (\bullet)$.
- What other matrix patterns are *linear*?
  [Füredi and Hajnal 1992; Keszegh 2009]

Reductions

Use *reductions* between matrix patterns:

- Rotation and reflection doesn't change anything, e.g.
  $\mathrm{Ex}(n, \begin{pmatrix} \bullet \\ \bullet \end{pmatrix}) = \mathrm{Ex}(n, \begin{pmatrix} \bullet & & \bullet \end{pmatrix})$,
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- Removing a 1-entry can only *reduce* $\mathrm{Ex}(n, P)$, e.g.
  $\mathrm{Ex}(n, \begin{pmatrix} \bullet & \bullet & \\ & \bullet & \bullet \end{pmatrix}) \geq \mathrm{Ex}(n, \begin{pmatrix} \bullet & & \\ & \bullet & \bullet \end{pmatrix})$

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Proposition. $\mathrm{Ex}(n, \begin{pmatrix} \bullet & & \\ \bullet & & \bullet \end{pmatrix}) \leq 2n$.

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- Idea: Adding a 1-entry at the “boundary” of P does not increase Ex very much.

Reductions

Lemma. [Füredi and Hajnal 1992] Let P be a matrix pattern. Consider a 1 in the topmost row of P, and add a 1 directly above it to obtain P'.
Then $\mathrm{Ex}(n, P) \leq \mathrm{Ex}(n, P') + n$.

- Example: $\begin{pmatrix} ? & ? & \bullet & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix} \to \begin{pmatrix} & & \bullet & \\ ? & ? & \bullet & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix}$

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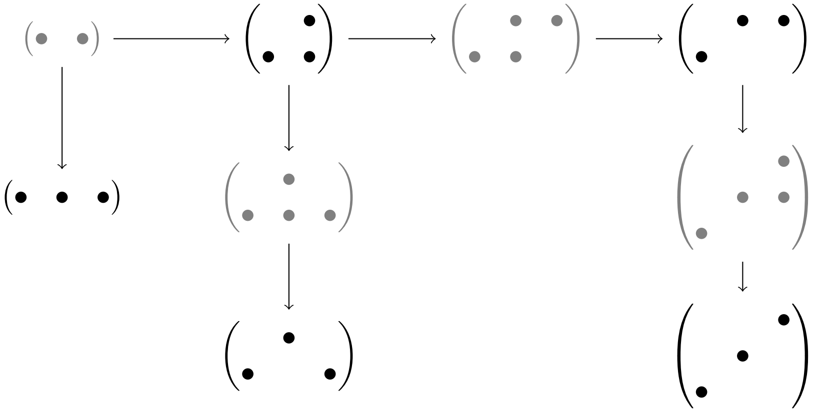
- Types of reduction: rotation/reflection, removing 1-entry, adding 1-entry at boundary.

Small linear patterns

All weight-3 patterns are linear.



Small linear patterns

- Reductions: at least 22 of the 37 weight-4 patterns are linear.
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- 3 more are linear. [Tardos 2005]
- The rest are non-linear: $\Theta(n\alpha(n))$, $\Theta(n \log n)$, or $\Theta(n^{3/2})$.

$$\begin{pmatrix} & & \bullet & \\ \bullet & & & \\ & \bullet & & \bullet \end{pmatrix} \ll \begin{pmatrix} \bullet & & \bullet & \\ & \bullet & & \bullet \end{pmatrix} \ll \begin{pmatrix} \bullet & \bullet & \\ \bullet & & \bullet \end{pmatrix} \ll \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$$

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- More reductions are known [Keszegh 2009]
- Weight-5 patterns not completely understood.

Large patterns

- *Light* patterns (one entry per column): $2^{\alpha^{\Theta(1)}(n)} n$.
  [Klazar 2001; Keszegh 2009]

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- *Light* patterns (one entry per column): $2^{\alpha^{\Theta(1)}(n)} n$.
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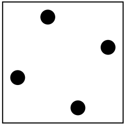
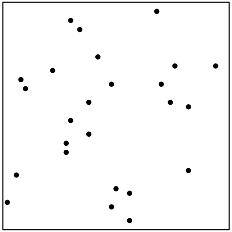
- *Füredi-Hajnal conjecture*: Permutation matrices (one entry per row and column) are linear.
  - Implies the *Stanley-Wilf conjecture* [Martin Klazar 2000]
  - Proven in 2004 by Marcus and Tardos.

Marcus-Tardos Theorem

Theorem. If P is a $k \times k$ permutation matrix, then $\text{Ex}(n, P) \in \mathcal{O}(n)$. [Marcus and Tardos 2004; Zeilberger 2003]



P



M

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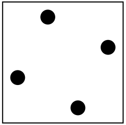

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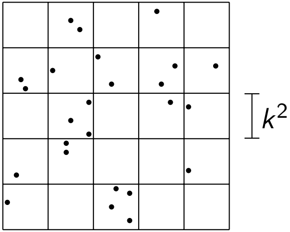
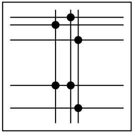


M

- Divide M into a grid of $k^2 \times k^2$ cells.

Marcus-Tardos Theorem – Heavy cells

- A heavy cell has $> (k-1)^2$ entries and is...
  - *high* if it has entries in $\geq k$ distinct rows;
  - *wide* if it has entries in $\geq k$ distinct columns.



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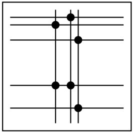
- High cells have $\binom{k^2}{k}$ choices for their k rows.



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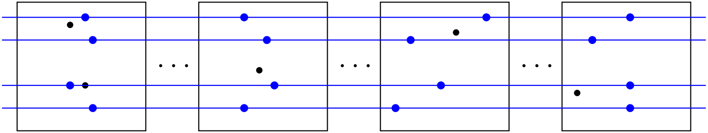

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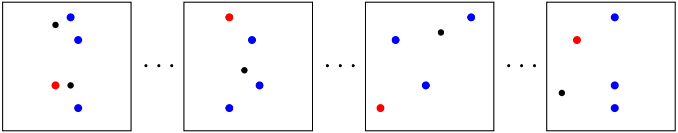
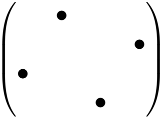


- Contains *every* $k \times k$ permutation matrix. Ex. 

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- Each grid column contains at most $k\binom{k^2}{k}$ wide cells.
- At most $2\frac{n}{k^2}k\binom{k^2}{k} \in \mathcal{O}(n)$ heavy cells
- At most $k^4 \in \mathcal{O}(1)$ entries per heavy cell:
- Heavy cells contribute $\mathcal{O}(n)$ entries.

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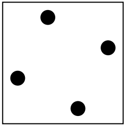
Light cells: $\leq (k-1)^2$ entries. Consider *all* non-empty cells:



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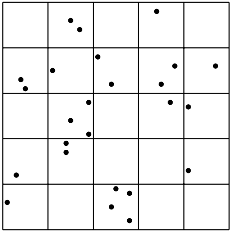

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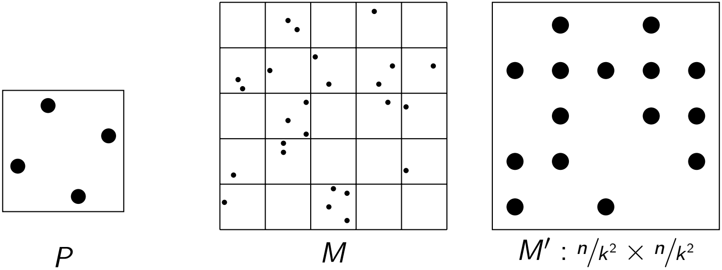

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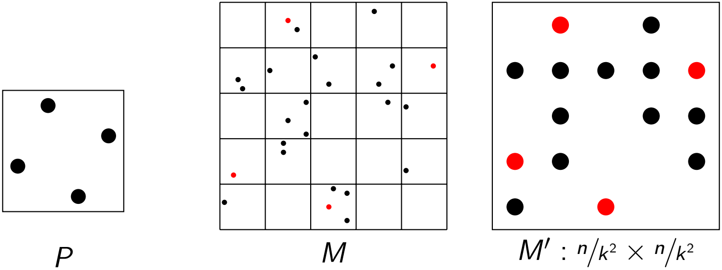

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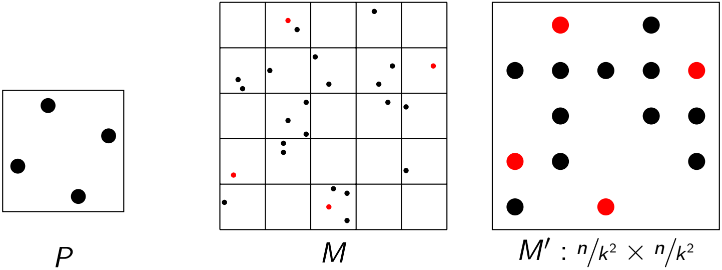

- At most $\mathrm{Ex}(n/k^2, P)$ non-empty cells
  $\implies$ Light cells contribute $(k-1)^2\,\mathrm{Ex}(n/k^2, P)$.

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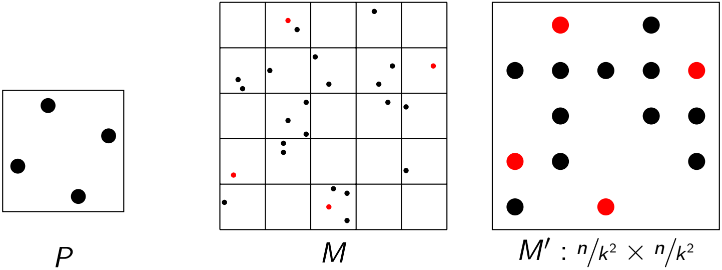

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  $\implies$ Light cells contribute $(k-1)^2\,\text{Ex}(n/k^2, P)$.
- $\text{Ex}(n, P) \leq (k-1)^2\,\text{Ex}(n/k^2, P) + \mathcal{O}(n)$
  $\implies$ $\text{Ex}(n, P) \in \mathcal{O}(n)$.

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- Algorithm for L_1-shortest paths with polgonal obstacles.
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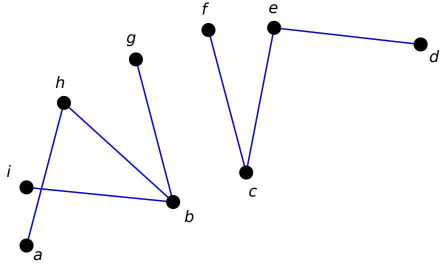

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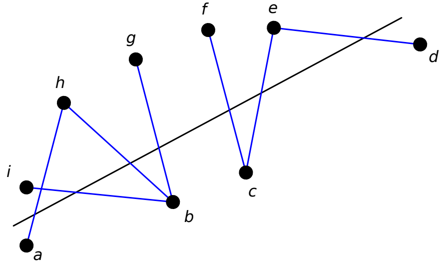
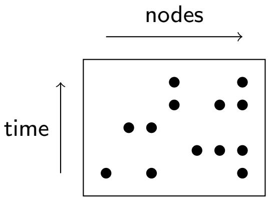


	a	b	c	d
e			•	•
f			•	
g		•		
h	•	•		
i		•		

Applications II

- Path compression in trees (e.g. union-find) [Pettie 2010a]



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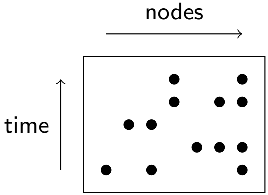

- Self-adjusting binary search trees (geometric BST model) [Pettie 2010a; Chalermsook et al. 2015; Kozma 2016]
- ... and more in discrete and computational geometry [Pettie 2010b]

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