Radon and Tverberg Numbers

Mittagsseminar 30 July 2020

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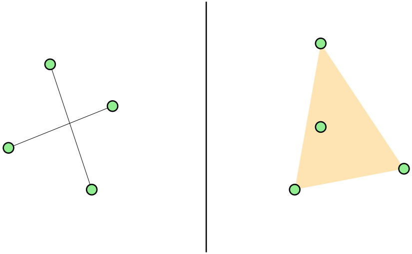

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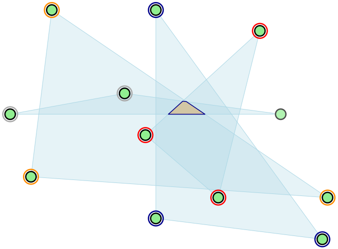

Tverberg number

The smallest number r_k such that every set of that size can be partitioned into k sets with intersecting convex hulls.
$r_k = (d + 1)(k - 1) + 1$ in $\mathbb{R}^d$.

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- a convex set $H \in C$ is a half-space if $X \setminus H \in C$
- space is separable if for every convex set $c \in C$ and $b \in X \setminus c$ there is a half-space H such that $c \subset H$ and $b \notin H$

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- Radon and Tverberg numbers for Convexity spaces?

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- for (V, C) (tree), $r = 4$
- for $(\mathbb{Z}^d, L^d)$ (lattice convex sets), $2^d \leq r = O(d2^d)$ [Onn, 1991]
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- what about r_k?

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- [Jamison, 1981] true for $r = 3$. r_k always exists when r exists,

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- upper bound $r_k = O(k^2 \log^2 k)$, constant depends on r

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Dömötör Pálvölgyi

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- if C is a family of sets in $\mathbb{R}^d$ with topological complexity at most b, then $r \leq f(b, d)$
- topological complexity depends on the Betti number of the intersections of elements of C
- eg. families of convex sets, good covers, pseudospheres, some families of semi-algebraics sets

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Fractional Helly Theorem [Holmsen and Lee, 2019]

Let (X, C) be a convexity space with radon number r. For each $\alpha \in (0, 1)$ there exists an integer $m(r)$ and $\beta(\alpha, r)$ such that

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- Dömötör's result uses this theorem
- Zuzana's result together with this gives a new result

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- ...but that is less than $\beta\binom{tk}{s}$, a contradiction (values of s, t matter here)

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- choosing the proper values for s, m, t, α makes sure everything works out

open questions

- algorithmic questions
- can $c(r) \leq r^{r^{r^{\log r}}}$ be made linear in r?
- extension to colorful Tverberg?