Proseminar Theoretische Informatik
An aperiodic set of 11 Wang tiles
Final Proof Summary/Explanation
Bashar Suleiman

Proposition 4: There are no words u, v s.t. u($T_{n+1} \circ T_n \circ T_{n+1} \circ T_n \circ T_{n+1}$)v

By section 6, Tn, when bordered by Tn+1, can be rewritten as concatenations of blocks: βγδ, εγδ, βδγεγδ, βγεγδ and βδεγδ.

By theorem 4 from section 6, $T_{n+3} \Longleftrightarrow T_{n+1} \circ T_n \circ T_{n+1}$.
Therefore: $T_{n+1} \circ T_n \circ T_{n+1} \circ T_n \circ T_{n+1} \Longleftrightarrow T_{n+3} \circ T_n \circ T_{n+1}$, T_n receives T_{n+3} outputs as input.

<u>Note:</u> The transducer pattern of T_n odd is being used in the following explanation.

The latter 4 transitions (εγδ, βδγεγδ, βγεγδ and βδεγδ) all contain εγδ, but T_{n+3} doesn't produce any output in which the markers “100” and “000” are close enough to be accepted as input for T_n:

- T_n: εαγαδ ⇒ 3 + g(n+2) - 3 + g(n+3) + 3 + g(n+2) - 3 + g(n+1) = g(n+5)
- T_{n+3}: Shortest path with similar markers is βαγ ⇒ 3 + g(n+4) + g(n+5) - 3 + g(n+5) =g(n+7)

This leaves only βγδ as a possibly valid input for T_n:

- T_n: δαβαγαδ ⇒ g(n+2) + g(n+2) + g(n+1) + g(n+2) + g(n+3) + g(n+2) + g(n+1) = g(n+6) (additions & subtractions of 3 omitted since they cancel each other out)
- T_{n+3} 000 ⇒ 000: g(n+4) + g(n+5) + g(n+5) = g(n+7)

Therefore, there exists no words u,v s.t. u($T_{n+1} \circ T_n \circ T_{n+1} \circ T_n \circ T_{n+1}$)v.

Proposition 5. Let n ≥ −2. Any tiling of the plane by T_D can be divided into strips of vertical width g(n),g(n+1) or g(n+2) so that each strip is a tiling by T_{u_n} ,$T_{u_{n+1}}$ or $T_{u_{n+2}}$. (Proof by induction)

Reminder, the word u_n is a sequence defined over [-2,∞) such that $u_{-2} = ε$,u_{-1} = a, u_0 = b,$u_{n+2} = u_n u_{n-1} u_n$.
First few values of u: ε,a,b,aa,bab,aabaa,babaabab,aabaababaabaa…
Initial conditions:

- For n = -2: Strips are tiled by T_a and T_b, which together make up T_D.
- For n = -1: Strips are tiled by T_a UT_b U T_{aa}, however T_{aa} is just the concatenation of T_a with itself. Therefore this case is equivalent to the first.

- For $n = 0$: Strips are tiled by $T_b \cup T_{aa} \cup T_{bab}$. This is proven to be true in Fact 5 from section 5.2.

I.A: Suppose the result holds for an arbitrary n. Then a tiling t by T_D can be divided into strips that correspond to tilings by T_{u_n} ,$T_{u_{n+1}}$ or $T_{u_{n+2}}$.

Since all 3 strips are components of a tiling by T_D, proposition 2 applies and the words in each row are elements of W. As proven in 5.2, for words u,v ∈ W, we can replace each strip T_{u_i} with a transducer of the defined transducer family T_i to obtain a tiling of the plane by $T_n \cup T_{n+1} \cup T_{n+2}$. Given the predefined inputs of these transducers, each row corresponding to the transducer Tn must be surrounded by rows corresponding to the transducer Tn+1.

But by the previous proposition there exists no words u,v s.t. $u(T_{n+1} \circ T_n \circ T_{n+1} \circ T_n \circ T_{n+1})v$, We can therefore replace each occurrence of 3 consecutive strips $T_{u_{n+1}}$, T_{u_n} ,$T_{u_{n+1}}$ by $T_{u_{n+3}}$ as no occurrences overlap. After doing this, no occurrence of T_{u_n} remains, which ends the proof.

Corollary 1. The Wang set $T_D = T_a \cup T_b$ is aperiodic.

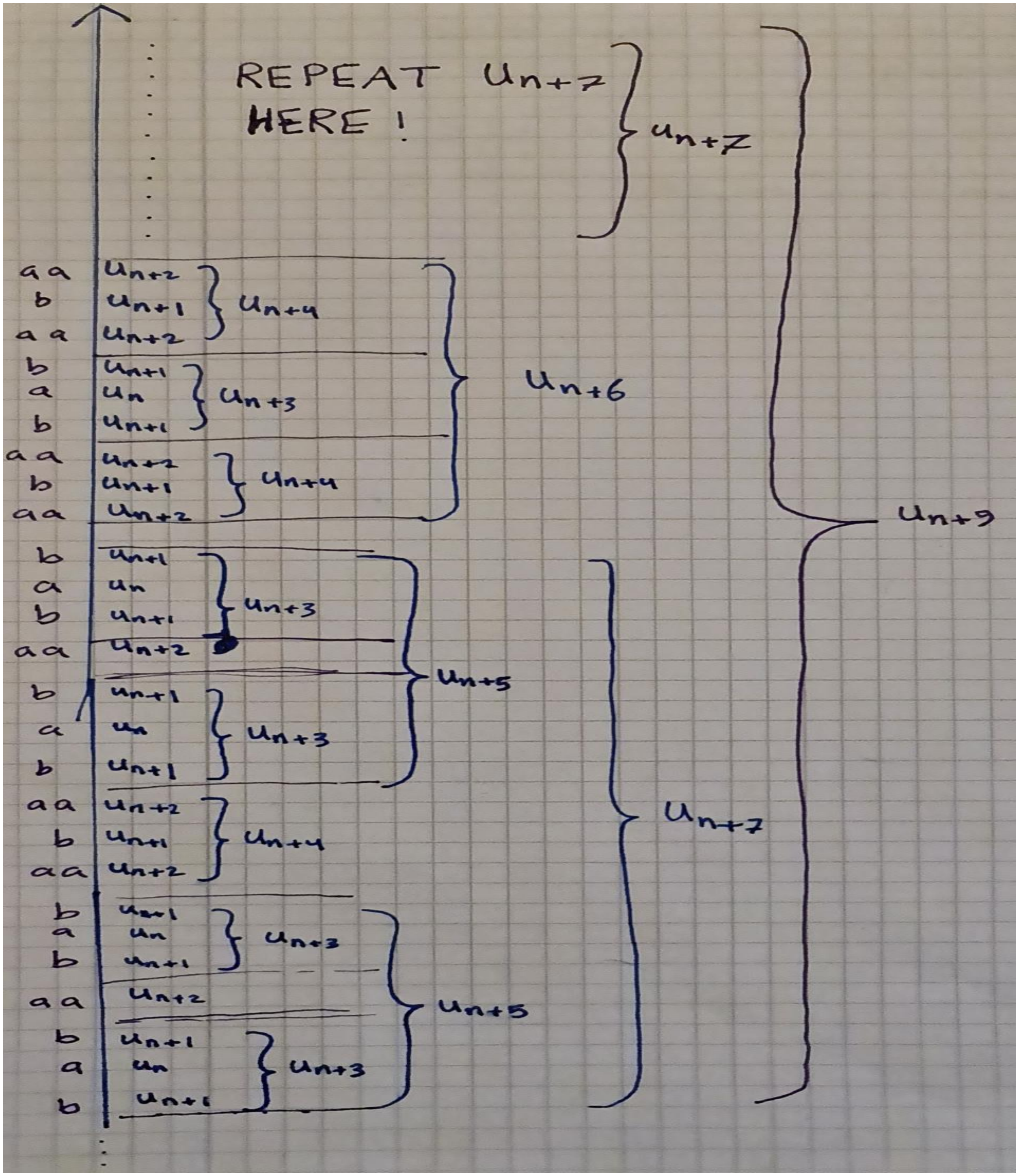
- First we prove that T_D tiles the plane: For all n, the transducer T_n contains a biinfinite path in its graph. This corresponds to a biinfinite run of the transducer over a strip of height n, which is equivalent to a wang tiling over a row of height n. But with the definition of the transducer family, we can create such a transducer for any $n \in [1,\infty)$. By compactness, there exists a tiling of the plane.
- Now to prove aperiodicity: Let v be the word over the alphabet {a,b} s.t. $v_i = a$ if the i-th row of the tiling corresponds to T_a and $v_i = b$ otherwise. By the previous proposition, any tiling by T_D can be decomposed into tilings by T_{u_n} ,$T_{u_{n+1}}$ or $T_{u_{n+2}}$ for all n, which implies that the word v can be written as a concatenation of u_n, u_{n+1} and u_{n+2}. The sequence of words u_n we defined is the sequence of singular factors of the Fibonacci word. Thus, v has the same set of factors as the aperiodic Fibonacci word.

Sturmian word: Cutting sequence for line of irrational slope.
Golden ratio: a+b/a = a/b := φ = *1.618…* (irrational number)
Fibonacci word: Sturmian word of slope $1/\varphi$

Corollary 2. The Wang set T is aperiodic. Furthermore, the set of words $u \in \{0,1\}^*$ s.t. the sequence of transducers Tu appears in a tiling of the plane is exactly the set of factors of sturmian words of slope $1/(\varphi +2)$, for φ the golden mean. The set of biinfinite words $u \in \{0,1\}^{\mathbb{Z}}$ s.t Tu which represents a valid tiling of the plane are exactly the sturmian words of slope $1/(\varphi +2)$.

Let ψ be the morphism a 7→ 10000,b 7→ 1000. The set of all words u ∈ {0,1} Z that can appear in a tiling of the whole plane are exactly the image by ψ of the sturmian words over the alphabet {a,b} of slope 1/φ. It is well known that the image of a sturmian word by ψ is again a sturmian word. But since sturmian words are by definition aperiodic, T is also aperiodic.

Demonstration of how T_D tiles the plane with three splits:

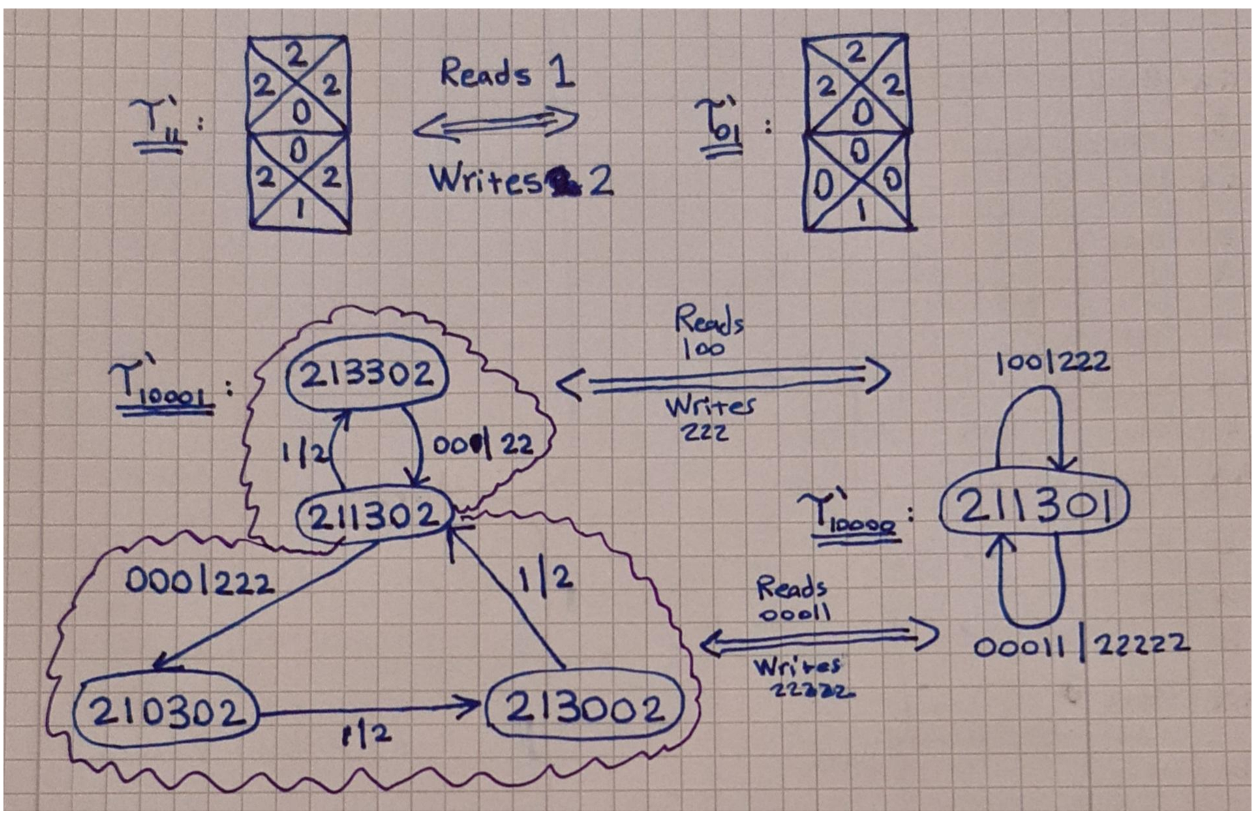


Section 7.2: In section 4, 3 candidates for aperiodicity were found. T' is the second candidate. It's almost identical to T, with the exception that every occurrence of "4" has been replaced with a "0". T' has the minimum number of colors for aperiodicity (4 colors) as well as the minimum number of tiles (11 tiles). Every tiling of T can be turned into a tiling of T', but can every tiling of T' be turned into a tiling of T?

We use similar steps to the ones we used to reduce T into T_D. First, we note that T' is the union of two Wang sets T'_0 and T'_1 of respectively 9 and 2 tiles.

Fact 6. The transducers $s(T'_{111})$, $s(T'_{101})$, $s(T'_{1001})$, $s(T'_{1000001})$, $s(T'_{10000001})$, $s(T'_{100000001})$, $s(T'_{000000000})$, $s(T'_{000011})$, $s(T'_{110000})$ and $s(T'_{1100011})$ are empty. Thus, if t is a tiling by T', then there exists a biinfinite binary word w ∈ {1000,10000,100011000, 100000000} Z such that t(x, y) ∈ T($T'_{w[y]}$) for every x, y ∈ Z. Let $T'_A = s(T'_{1000} \cup T'_{10000} \cup T'_{100000000} \cup T'_{100011000})$. As before, T'_A has unused transitions (those which write 2 or 3). Once deleted, and then once having deleted states which cannot appear in a tiling of a row, we obtain T'_B.

Proposition 6: T'_{11} is isomorphic to a subset of T'_{01}, and T'_{100000} is isomorphic to a subset of T'_{100001}.



Corollary 3: **Tc and Td are both isomorphic to a subset of Ta ◦Tb**

$T_c \Longleftrightarrow T'_{\underline{100000}000}$ can be decomposed into a concatenation $T_b \circ T_a$ by replacing the underline part with 10001 in accordance with the previous section.
$T_c \Longleftrightarrow T'_{000\underline{11}000}$ can be decomposed into a concatenation $T_b \circ T_a$ by replacing the underline part with 01 in accordance with the previous section.

Theorem 5. The Wang set T' is aperiodic.
Since every tiling by T_B can be turned into a tiling by T'_B, and every tiling by T'_B can be turned into a tiling by T_B in accordance with corollary 3, T' must be aperiodic.

Section 7.3: The final candidate for aperiodicity is the wang set T''. It is again very similar to T, with the exception that the tile (1,1,4,2) has been replaced with the tile (3,3,4,2). The reduction of T'' to T''_c is analogous to the reduction of T_c, and yields a very similar transducer graph. However, T''_{1000} (resp. 10000) does not produce the exact same outputs as T_{1000} (resp. 10000). To achieve the same behavior as T, we concatenate each tile from T'' with a shift transducer. This transducer will shift the results one position to the left when and changes state at nodes: 21130, 23130, 23030, 21330.